The chain rule and inverse functions...

RULES So FAR.

1. $\frac{d}{dx}[c f(x)] = c \frac{df}{dx}$ 2. $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$

3. $\frac{d}{dx}\left[x^{n}\right] = n x^{n-1}$ 4. $\frac{d}{dx} a^{x} = \ln(a) a^{x}$ or $\frac{d}{dx} e^{x} = e^{x}$

5. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = \frac{df}{dx} \cdot g(x) + \frac{dg}{dx} \cdot f(x)$

6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df}{dx} \cdot g(x) - \frac{dg}{dx} \cdot f(x)}{g^2(x)}$

7. $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$

8. $\frac{d}{dx}\left[\tan(x)\right] = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x$

Today we combine what we know about the inverse and these rules to find our last few rules.

What is the Inverse... $y = f(x)$ then $f^{-1}(y) = x$ or $f^{-1}(x) = g(x)$

the new function

"solving" for the dependent variable in terms of the independent variable.

Ex $y = x^3 - 1$ then solving for x... $x^3 = y + 1$ and $x = (y+1)^{1/3}$

so $f^{-1}(x) = (x+1)^{1/3}$

Ex $y = e^x$ find the inverse... $\ln(y) = \ln(e^x) = x$

so $f^{-1}(x) = \ln(x)$

Lets use this idea to prove some more rules...

Ex One we have been using but never really PROVED

$\frac{d}{dx} x^n = n x^{n-1}$ when n is NOT an integer.

Consider $f(x) = x^{1/2}$ we have been doing..

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}}$

let's prove this actually works... USE THE INVERSE!

1. Solve for Inverse $y = f(x) = x^{1/2}$ then $y^2 = x$

so the inverse is $(f(x))^2 = x$.

2. Take the derivative of both sides.

$$\frac{d}{dx}(f(x))^2 = \frac{d}{dx} x$$

Chain Rule!

3. Solve for $f'(x)$ and plug in.

$2f(x) \cdot f'(x) = 1$ so $f'(x) = \frac{1}{2f(x)} = \frac{1}{2x^{1/2}}$ ✓

IT WORKS!

Now lets see how this works for derivatives we don't already know...

Ex $f(x) = \ln(x)$ what is $f'(x)$? hmmm
we don't have a rule
for this yet !

1. Consider the inverse:

$$e^{f(x)} = e^{\ln(x)} = x$$ I know $\frac{d}{dx} e^x = e^x$!

2. Take the derivative

$$\frac{d}{dx} e^{f(x)} = \frac{d}{dx} x$$

Chain Rule.

$$e^{f(x)} \cdot f'(x) = 1$$

3. Solve for $f'(x)$ $\quad f'(x) = \frac{1}{e^{f(x)}}$ but $f(x) = \ln(x)$

so $f'(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$ ✓

NEW RULE $\quad \frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$ note $x > 0$ in $\ln(x)$.

I prefer this !!!

EX $f(x) = \arctan(x) = \tan^{-1}(x)$

what is $\frac{d}{dx}\left[\arctan(x)\right]$? YOU TRY !!

1 Consider the inverse: $\tan(f(x)) = \tan(\arctan(x)) = x$

2. Derivative: $\frac{d}{dx}\tan(f(x)) = \frac{d}{dx}(x)$

$$\frac{1}{\cos^2(f(x))} \cdot f'(x) = 1$$

3. Solve: $f'(x) = \cos^2(f(x)) = \cos^2(\arctan(x))$ ✓

SNEAKY! ... we can simplify this! $\sin^2 x + \cos^2 x = 1$ or $\tan^2 x + 1 = \frac{1}{\cos^2 x}$

so $\cos^2(x) = \frac{1}{\tan^2 x + 1} \Rightarrow \cos^2(\arctan(x)) = \frac{1}{\tan^2(\arctan(x)) + 1} = \frac{1}{x^2+1}$

$$\boxed{\text{NEW RULE} \qquad \frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}}$$

__Ex__ $\quad f(x) = \arcsin(x) \quad$ YOU TRY !!

what is $f'(x) = \frac{d}{dx}\left[\arcsin(x)\right]$

$$\sin(f(x)) = x$$

$$\frac{d}{dx}\left[\sin(f(x))\right] = 1$$

$$\cos(f(x)) \cdot f'(x) = 1 \quad \text{so} \quad f'(x) = \frac{1}{\cos(\arcsin(x))}$$

simplify we would like $\sin(\arcsin(x)) = x$ ...

$$\sin^2 x + \cos^2 x = 1$$

$$\cos(x) = \sqrt{1 - \sin^2 x}$$

so $f'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\text{NEW RULE} \qquad \frac{d}{dx}\left[\arcsin(x)\right] = \frac{1}{\sqrt{1-x^2}}}$$

__Ex__ Find $f'(x)$ if $f(x) = \arcsin(\tan(x))$

could us the inverse OR chain rule directly.

$$f'(x) = \frac{1}{\sqrt{1 - \tan^2 x}} \cdot \frac{d}{dx}\left[\tan(x)\right] = \frac{1}{\sqrt{1 - \tan^2(x)}} \cdot \frac{1}{\cos^2 x}$$