DAY 16

Moving Forward... Once we have proved a functions derivative we can use that RULE !

Keep this list handy... <u>MEMORIZE THEM</u>

Going forward we are using the rules to build more interesting mathematics.

Today... Chapter 3.7 Implicit Functions.

EXPLICIT FUNCTION — can be written as

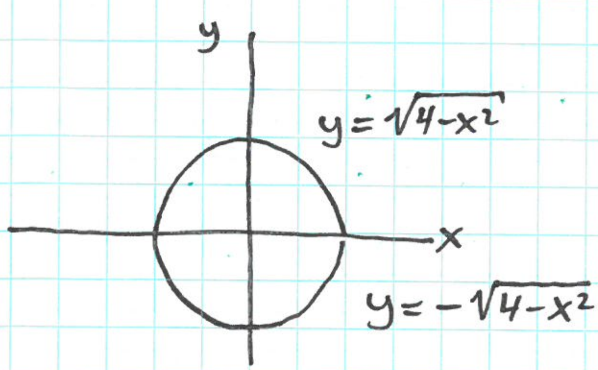
$y = f(x)$ we can solve for just one y expressed only in terms of x.

IMPLICIT FUNCTION — cannot be written so simply.

<u>Ex</u> Equation for a circle:

$$x^2 + y^2 = 4$$

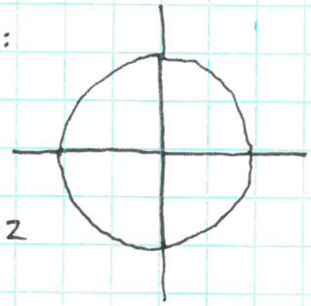
try to solve for y: $y = \pm\sqrt{4-x^2}$



Not really a function! TWO x-values.

So lets think about the circle:

$$x^2 + y^2 = 4$$



circle r=2

Does this have tangent lines at every point?
YES! We can still talk about slope

We want to be able to take the derivative of these!

DERIVATIVE OF IMPLICIT FUNCTIONS.

$$x^2 + y^2 = 4 \longrightarrow x^2 + [f(x)]^2 = 4 \qquad y = f(x)$$

we still think of y as a function of x.
we just can't solve for it explicitly.

1. Take derivative of both sides...

$$\frac{d}{dx}\left[x^2 + [f(x)]^2\right] = \frac{d}{dx} 4$$

$$2x + \frac{d}{dx}[f(x)]^2 = 0$$

this needs the chain rule!

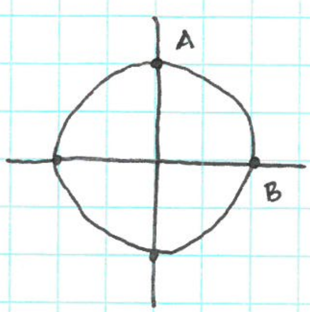
$$2x + 2f(x)f'(x) = 0$$

2. Solve for $f'(x)$ ...

$$2f(x)f'(x) = -2x \qquad f'(x) = \frac{-x}{\cancel{2}f(x)}$$

so $\dfrac{dy}{dx} = \dfrac{-x}{\cancel{2}y}$

Does this make sense... the slope of a tangent line is $-x/x$?



@ the point A
(0,2) the slope is zero

$$\frac{dy}{dx} = -\frac{0}{2} = 0$$

@ the point B
(2,0) the slope is undefined

$$\frac{dy}{dx} = -\frac{2}{0} = \text{undefined.}$$

The derivative of an implicit functions allows us to answer lots of questions about complicated implicit functions!

EX $y^2 + xy = x$ PLOT ON DESMOS.

1. Eqn of tangent line @ $y=3$
2. where is the slope zero?
3. Where is the slope undefined?

FIRST FIND $\frac{dy}{dx}$ ... $y^2 + xy = x \xrightarrow[y=f(x)]{} [f(x)]^2 + xf(x) = x$

Derivative of both sides... $\frac{d}{dx}\left[[f(x)]^2 + xf(x)\right] = \frac{d}{dx}x$ you will always need the chain rule!

$$2f(x)f'(x) + f(x) + xf'(x) = 1$$

Solve for $f'(x)$

$$2f(x)f'(x) + xf'(x) = 1 - f(x)$$

$$[2f(x) + x]f'(x) = 1 - f(x)$$

$$f'(x) = \frac{1-f(x)}{2f(x)+x} \quad \text{or} \quad \frac{dy}{dx} = \frac{1-y}{2y+x}$$

1. TANGENT LINE : $y = mx + b$
I need the slope and a point!

FIND THE POINT... if $y=3$ then plugging into our original implicit function I find

$$(3)^2 + x(3) = x \longrightarrow 9 + 3x = x$$
then solve for x

$2x = -9$
$x = -9/2$
The point is $\boxed{(-\frac{9}{2}, 3)}$

FIND THE SLOPE... $\frac{dy}{dx} = \frac{1-y}{2y+x}$

$$\frac{dy}{dx} = \frac{1-3}{2(3)-9/2} = \frac{-2}{6-9/2} = \frac{-2}{\frac{12}{2}-\frac{9}{2}} = \frac{-2}{\frac{3}{2}} = \boxed{\frac{-4}{3}}$$

FIND THE LINE $y = -\frac{4}{3}x + b$

$3 = -\frac{4}{3}\left(-\frac{9}{2}\right) + b$

$3 = -2 \cdot -3 + b$

$3 = 6 + b$ or $b = -3$

$\boxed{y = -\frac{4}{3}x - 3}$

2. Where is the slope zero?

$\frac{dy}{dx} = \frac{1-y}{2y+x}$ zero slope means that the numerator is zero

IS $y=1$ allowed in our implicit f

$1-y=0$ or when $\boxed{y=1}$

but $1^2 + x = x$ or $1=0$ $\boxed{\text{NOT TRUE}}$

so the slope is NOT REALLY EVER ZERO!

3. Where is the slope undefined?

$\frac{dy}{dx} = \frac{1-y}{2y+x}$ undefined slope means that the denominator is zero

$2y+x=0$ or when $x=-2y$ $\boxed{y = -\frac{x}{2}}$

But these points also need to be on the curve!

plug into the eqn... plug in $x=-2y$

$y^2 + xy = x$

$y^2 - 2y^2 = -2y$

$-y^2 = -2y$

$y^2 - 2y = 0$

$y(y-2) = 0$ when $y=0$ or $y=2$

Always check that the curve agrees!!

TWO POINTS.

undefined at $(0,0)$ and $(-4,2)$

EX $\quad xy + y^3 = 1$

Lets work on this together!

1. Find the derivative $\frac{dy}{dx}$
2. Eqn of a tangent line at $y=1$
3. Is the slope ever zero or undefined?

1. $\frac{d}{dx}\left[xy + y^3\right] = \frac{d}{dx} 1 \qquad y + xy' + 3y^2y' = 0$

$$(x+3y^2)y' = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x+3y^2}}$$

2. TANGENT LINE. $\quad y = mx + b$

where on the cure is $y=1$

FIND THE POINT(s) $\quad$ if $y=1$ then

$$x + 1 = 1 \quad \text{or} \quad x = 0$$

the point $\boxed{(0,1)}$

FIND THE SLOPE $\quad \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-1}{0+3} = \boxed{-1/3}$

plug in the point

PLUG IN: $\quad y = -1/3\, x + b \qquad 1 = 0 + b \quad b = 1$

$$\boxed{y = -1/3\, x + 1}$$

3. Vertical or Horizontal tangents?

$\frac{dy}{dx} = 0$ when numerator: $-y = 0 \ldots$

zero slope $\Rightarrow \quad -y = 0$ or $y = 0$ $\quad$ if $y=0$ then

CHECK — $0 + 0 = 1 \quad 0 = 1$ $\quad$ $y=0$ is not part of curve.

NOT TRUE — slope is never zero!

UNDEFINED SLOPE $\Rightarrow \quad x + 3y^2 = 0$ or $x = -3y^2$

CHECK $\quad -3y^2 \cdot y + y^3 = 1 \qquad -3y^3 + y^3 = 1 \qquad -2y^3 = 1$

when $y = \frac{-1}{\sqrt[3]{4}}$ we have $x = 3y^2$ and slope undefined. $\qquad y = \frac{-1}{\sqrt[3]{2}}$ $\quad$ yes it does become undefined!

For each of the problems below do the following:

1. Find all points on the curve with the given y value.

2. Graph the function on Desmos.com and check that these points are on the curve. Do you expect slopes to be positive or negative at these points?

3. Find the derivative $\frac{dy}{dx}$.

4. Find the equation of the tangent line for each of the points you found in part 1.

5. Check your tangent lines by adding them to the graph you made in part 2.

1. $x^2 + xy - y^3 = xy$ and $y = 1$
2. $\sin(xy) = 2x + 5$ and $y = 0$
3. $x^2y - 2y + 5 = 0$ and $y = 5$
   For number 3 ONLY:
   6. Discuss any locations where the slope is zero (aka where the tangent line is horizontal).
   7. Discuss any locations where the slope is undefined (aka where the tangent line is vertical).