We will Skip 3.8 Hyperbolic Functions:

$$\sinh(x) = \frac{e^x + e^{-x}}{2} \qquad \cosh(x) = \frac{e^x - e^{-x}}{2}$$

Used often in Engineering...
just need the definitions.

From today forward we are going to be <u>using</u> the derivative to solve "real" problems

TODAY — Linear Approximations and the derivative.
* USED often in Physics and other Sciences!!

Linear approximations are used to approximate functions at a hard point using what we know about the function at an easy point.

<u>Ex</u> $f(x) = y = (x + 1/2)^2$

$f(1/2) = (1)^2 = 1$ easy!
what is $f(.5237)$? hard!

Lets look at a graph: If we zoom in... how could we guess $f(.5237)$ without a calculator... STRAIGHT LINE GOES THROUG $(\frac{1}{2}, 1)$ SAME SLOPE!

TANGENT LINE!

<u>Ex</u> $f(x) = \sin(5x)$

$f(0) = \sin(0) = 0$ Easy
what is $f(.1)$? Hard!

well lets try a <u>TANGENT</u> <u>LINE</u> at the <u>EASY</u> point

... Find a tangent line to $y = \sin(5x)$ at the point $x = 0$

$$m = y'\Big|_{x=0} = \cos(5x)\cdot 5\Big|_{x=0} = \cos(0)\cdot 5 = 5$$

$y = 5x + b$ point: when $x = 0$ $y = \sin(0) = 0$ $(0,0)$

TANGENT LINE: $\boxed{y = 5x}$

NOW What is $f(.1) = \sin(5 \cdot .1) \approx \sin(.5) \sim 5(.1) = .5$
$x = .1$

REAL CALCULATOR ANSWER $\sin(.5) = .4794255...$
we were close.

GRAPH on DESMOS.

How accurate are we? Is this always good? $f(10) \sim$?

WHAT ARE THE STEPS TO APPROXIMATING?

Approximate $f(x)$ near $x = a$ ... tangent line approximation.

1. Find the point ... If $x = a$ then $y = f(a)$ plug in.

$$(a, f(a))$$

2. Find the slope at $x = a$

$$m = \frac{dy}{dx}\Big|_{x=a} = f'(a)$$

3 Tangent line equation $y = mx + b = f'(a)x + b$

solving for b ... $f(a) = f'(a)a + b$ $b = f(a) - f'(a)a$

so $y = f'(a)x + f(a) - f'(a)a$

$$\boxed{y = f'(a)[x - a] + f(a)}$$

This is so cool ... a FORMULA
for a TANGENT LINE APPROXIMA

TANGENT LINE APPROXIMATION (aka local linearization)

Suppose f is differentiable at a. Then for values of x near a, the tangent line approximation is given by

$$f(x) \approx f(a) + f'(a)[x-a]$$

and the ERROR is defined by

$$E(x) = \underbrace{f(x)}_{\text{REAL}} - \underbrace{\left[f(a) - f'(a)[x-a]\right]}_{\text{APPROXIMATE}}$$

this can be approximated by

$$E(x) \approx \frac{f''(a)}{2}[x-a]^2$$

This makes sense ... on $\sin(5x)$ graph where is the highest curvature?
How well would a tangent line approximate points near peaks and troughs?

Ex What is the local linearization of $f(x) = e^{kx}$ near $x=0$

Tangent line Approx: $f(x) \approx f(a) + f'(a)[x-a]$ $\quad a=0$

$$e^{kx} \approx e^{k(0)} + \underbrace{ke^{k(0)}}_{\text{derivative.}}[x-0] = 1 + kx$$

Using this with $k=1$ approximate $f(.1)$

$f(x) = e^x$ then $f(x) \approx 1+x$ near $x=0$

so $f(.1) = 1 + .1 = 1.1$

Approximate the ERROR in this estimate.

$$E(.1) \approx \frac{f''(0)}{2}[.1-0]^2 = \frac{e^0}{2}[.1]^2 = \frac{1}{2}[.01] = .005$$

Compare with Calculator answer $f(.1) = e^{.1} = 1.10517\ldots$ pretty close!

Ok → So far we are just approximating things that we can already calculate — so where is this used?

Ex. The acceleration of gravity:

Often in physics you will hear that the acceleration of gravity is constant:

$$g = 9.8 \ m/s^2 \text{ on earth.}$$

If we really model the acceleration of gravity we get:

Law of Gravitational Pulls.

$$g(r) = \frac{GM}{r^2}$$

M = mass of Earth
G = Universal gravitational constant
r = distance from the earths center.

But this is not constant!
How can they get away with that?

Lets linearize this $g(r)$ near $r = R_E$ = radius of the earth.

- We are approximating this near the earths surface.

1. Write down approximation $f(x) \approx f(a) + f'(a)(x-a)$

so $\underset{\substack{\text{near} \\ r=R_E}}{g(r)} \approx g(R_E) + f'(R_E)(r - R_E)$

2. Find all the pieces $g(R_E) = \frac{GM}{R_E^2}$ $\quad g'(r) = \frac{-2GM}{r^3}$ $\quad g'(R_E) = \frac{-2GM}{R_E^3}$

3. Put it all together so $g(r) \approx \frac{GM}{R_E^2} - \frac{2GM}{R_E^3}(r - R_E)$

Now ~~we~~ look at this and ~~says~~ Well the earths radius is very large so something with $\frac{1}{R_E^3}$ is really small!

So we will use the approximation:

$$g(r) \approx \frac{GM}{R_E^2} \equiv 9.81 \ m/s^2$$

Now lets say you want to throw a water balloon from your dorm window and hit a passing student...

would you rather assume gravity is constant? or recalculate it at every instant as that balloon is getting closer to the center of the earth?

YAY – math makes life easier!!

Would it make sense to use this approximation far from the earths surface?

Consider the Error

$$g''(r) = \frac{6GM}{r^4} \qquad g''(R_E) = \frac{6GM}{R_E^4}$$

$$E(r) \sim \frac{3GM}{R_E^4}(r - R_E)^2$$

constant

as long as r is close to R_E the error remains small