[redacted]

TALK ABOUT SURVEY

4.4 Optimization, Geometry and Modeling.

Mathematical Modeling – taking a problem from the real world and understanding it mathematically.

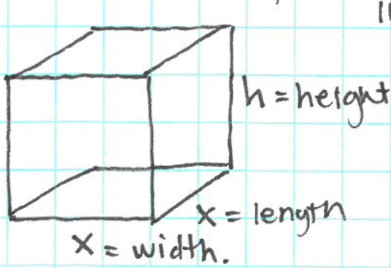
HINTS FOR SUCCESSFUL MODELING

1. Draw pictures and label important things.
2. Reword the problem so YOU understand it.
   EX. increasing function $\rightarrow$ positive slope or derivative > 0
3. Max or Min $\Rightarrow$ looking for critical points.
4. Write in words how you would solve the problem.

EX A closed rectangular box with a square bottom is to hold 8cm^3 of a product. What dimensions of the box minimize surface area and thus the material cost?

1 Draw a picture.

Label Things!



IMPORTANT WORDS
closed box ✓
rectangular ✓
square bottom ✓
Draw this.

2. What do we know...
needs to hold 8cm^3
so Volume $= 8\text{cm}^3$
OR
$$\boxed{x^2 h = 8\text{cm}^3}$$
TO START
write equation for what is given

3. What do we want to minimize or maximize
Surface Area = top + bottom + sides.

MIN $\rightarrow$ Surface Area $= 2x^2 + 4xh$

TOP AND BOTTOM SIDES.

Then Use that!

write eqn for what we want to know.

4. Step back... what equations do I have? can I put them together into one?

$\boxed{x^2h = 8}$ and $\boxed{SA = 2x^2 + 4xh}$

solve for one var.

$h = \dfrac{8}{x^2}$ → plug in

$SA = 2x^2 + 4x \cdot \dfrac{8}{x^2}$

$SA = 2x^2 + \dfrac{32}{x}$

Now I have SA in terms of ONE VARIABLE $SA = f(x)$

5. Min or Max: MINIMIZE THIS! use critical points.

$$\frac{d\,SA}{dx} = 4x - \frac{32}{x^2} = 0$$

$4x = \dfrac{32}{x^2}$

$4x^3 = 32$

$x^3 = 8$

$x = 2$ This is a critical point.

IS IT A MIN? Always Check!

FIRST DERIV TEST

$x=1$	$x=2$	$x=3$
neg	0	pos

It is a min!

-OR- SECOND DERIV TEST $\dfrac{d^2SA}{dx^2} = 4 + \dfrac{64}{x}$ plug in 2... >0 MIN

6. Plug back in to get all the dimensions...

$\boxed{x=2}$ and $h = \dfrac{8}{x^2} = \dfrac{8}{4} = \boxed{2}$

Good to Check...

so $\boxed{x = h = 2}$ minimizes the surface area while keeping $8\,cm^3$ volume.

Is this global min?

$x \geq 0$ $SA \to \infty$ as $x \to 0^+$

$SA \to \infty$ as $x \to \infty$

SO YES GLOBAL MIN.

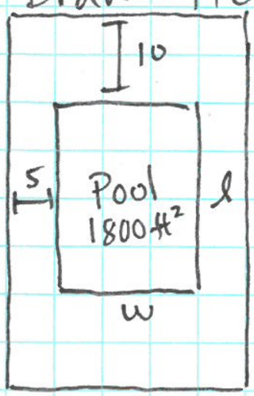
NOTE: Each modeling problem is different... I can't do them all in class... but take a deep breath and follow the steps and we should be OK.

Remember to have patience in solving hard problems its ok to not know how but to just try something

Don't be afraid to use old geometry rules! or look them up

Ex A rectangular swimming pool is to be built with an area of $1800\ ft^2$. The owner wants 5-foot wide decks along the sides and 10 foot decks at the two ends. What is the smallest piece of land on which this pool can be built.

1 Draw Pictures



Important Words
Area of Pool $= 1800\ ft^2$
5 ft decks
10 ft decks
Rectangular

2. What do we know

$lw = 1800$

3. What do we want to minimize or maximize?

AREA OF LAND = AREA OF POOL + AREA OF DECK
OR
= (TOTAL WIDTH)(TOTAL LENGTH)

$$A = (w+10)(l+20) = wl + 10l + 20w + 200$$

4. Step back and put things together

$lw = 1800$ $\qquad$ $A = wl + 10l + 20w + 200$

solve for one variable

$w = \frac{1800}{l}$ $\qquad$ plug in

$$= 1800 + 10l + \frac{20 \cdot 1800}{l} + 200$$

$$A = 2000 + 10l + \frac{36000}{l}$$

5. MIN OR MAX

$$\frac{dA}{dl} = 10 - \frac{36000}{l^2} = 0 \qquad 10 = \frac{36000}{l^2}$$

$$l^2 = 3600 \qquad l = \pm 60\ ft.$$

well $l = -60$ ft doesn't make sense

$l=59$	$l=60$	$l=61$
neg	0	pos
↘	—	↗

it is a min!

6. Plug back in to get all the dimensions.

$$w = \frac{1800}{l} = \frac{1800}{60} = 30 \text{ ft.} \qquad \begin{array}{l} w=30 \\ l=60 \end{array}$$

The Plot of Land must be $w+10 = 40$
$l+20 = 80$

or $40 \times 80 = \boxed{3200 \text{ ft}^2}$

Is it global?
$\lim_{x\to 0^+} A = \infty$
$\lim_{x\to \infty} A = \infty$
YES GLOBAL MIN.

Always double check that you answered the original question!

EX Pure Geometry ... The Hypotenuse of a right triangle has one end at the orgin and the other on the curve $y = x^2e^{-3x}$ with $x>0$. One of the sides is on the x-axis and the other is parallel to the y-axis. Find the maximum area of this triangle.

1. Draw Pictures.

IMPORTANT WORDS.
Hypotenus from (0,0) to $y=f(x)$
Right triangle
one side on x
other $\parallel$ to y.

2. What do we know?

$b = x$
$h = f(x) = x^2e^{-3x}$

3. What are we maximizing?

$$\text{AREA} = \tfrac{1}{2}bh = \tfrac{1}{2}x \cdot x^2e^{-3x} = \tfrac{1}{2}x^3e^{-3x}.$$

4. step back ... $A = \frac{1}{2}x^3e^{-3x}$

and we want to maximize!

OK we can do this!

5. MAX or MIN

$$\frac{dA}{dx} = \frac{1}{2}3x^2e^{-3x} - \frac{3}{2}x^3e^{-3x}$$

$$= \frac{3}{2}e^{-3x}(x^2 - x^3) = 0$$

so $(x^2 - x^3) = 0$

$x^2(1-x) = 0$ $\quad x=0$ or $x=1$

$x=-1/2$	$x=0$	$x=1/2$
pos	0	pos

NEITHER.

$x=1/2$	$x=1$	$x=2$
pos	0	neg

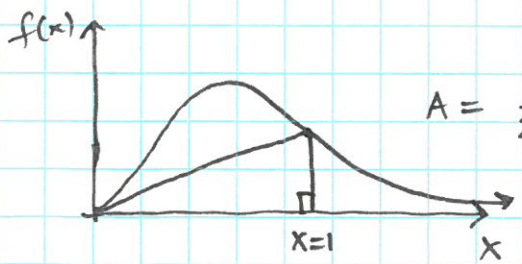
MAX

Also $x=0 \Rightarrow$ zero Area!

6. Plug back in to get all dimensions.

$$A = \frac{1}{2}x^3e^{-3x} = \frac{1}{2}1^3e^{-3} = \frac{1}{2}e^{-3} = \boxed{\frac{1}{2e^3}}$$

This is the Maximum Area.



$A = \frac{1}{2e^3}$

Is this the global max

$x \geq 0$ $\quad A=0$ when $x=0$

$\lim_{x\to\infty} A = 0$

so yes it is global.

Ex UPPER AND LOWER BOUNDS ... It is often important to know how big or small a function could get.

Ex You are building a machine with a part on a spring. You find experimentally that it's position seems to follow the eqn

$$y = e^{-t}\cos(t)$$

what is its greatist displacement for $t \geq 0$

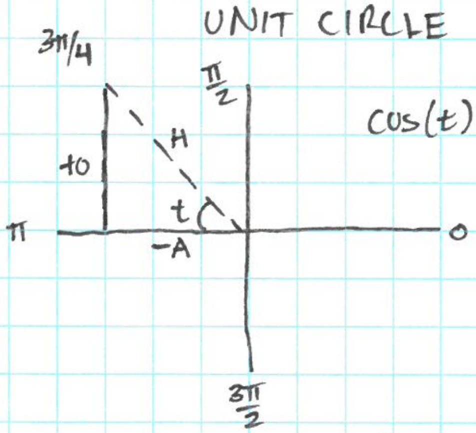
In other words → how big of a space do I have to carve out to fit this spring in my machine.

We just need to find the global max and min of our displacement function.

$$\frac{dy}{dt} = -e^{-t}\cos(t) \mp e^{-t}\sin(t)$$
$$= -e^{-t}(\cos(t) + \sin(t)) = 0$$

this is when $\cos(t) = -\sin(t)$

ONE WAY
UNIT CIRCLE



$\cos(t) = \frac{-A}{H}$ $\quad \sin(t) = \frac{0}{H}$

$$\frac{-A}{H} = \frac{-0}{H}$$

$\Rightarrow$ $A = 0$

so $t = 45° = \frac{3\pi}{4}$

ANOTHER WAY
DIVIDE BY $\cos(t)$

$$1 = -\frac{\sin(t)}{\cos(t)}$$

$\tan(t) = -1$

then

$t = \frac{3\pi}{4}$

or $t = 7\pi/4$

There are other angles that would work but e^{-t} decreases so quickly that the first angle is our best bet...

Use a graph if needed!

$t = \frac{3\pi}{4}$ is a critical point

$$y\left(\tfrac{3\pi}{4}\right) = e^{-3\pi/4}\cos\left(\tfrac{3\pi}{4}\right) \sim -0.067$$

check $t = \frac{7\pi}{4}$ $\quad y\left(\frac{7\pi}{4}\right) = e^{-7\pi/4}\cos\left(\frac{7\pi}{4}\right) \sim$ much smaller

Check Edges

$t = 0$ $\quad y(0) = 1$ $\qquad$ So upper bound $y = 1$

$\lim_{t \to \infty} e^{-t}\cos(t) = 0$ $\qquad$ lower bound $y = -0.$

BEST POSSIBLE BOUNDS