4.7 L'Hopitals Rule.

In many problems in calculus we are asked to consider the limit of a function. We should be pretty good at this!

Ex $\lim\limits_{x\to\infty} e^{-x} + 3$ we know that e^{-x} is decaying to zero so...

$$= 0 + 3 = 3$$

Ex $\lim\limits_{x\to\infty} \dfrac{x-5}{5+2x^2}$ sometimes we can use algebra to simplify...

$$\frac{x-5}{5+2x^2} = \frac{x-5}{x^2\left(5/x^2+2\right)} = \frac{1/x - 5/x^2}{5/x^2+2}$$

Then using rules for limits:

$$\lim_{x\to\infty} \frac{x-5}{5+2x^2} = \frac{\lim\limits_{x\to\infty} 1/x - 5/x^2}{\lim\limits_{x\to\infty} 5/x^2+2} = \frac{0-0}{0+2} = \frac{0}{2} = 0 \checkmark$$

But what about problems like:

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} \quad \ldots \quad \frac{\lim\limits_{x\to 0} e^{2x}-1}{\lim\limits_{x\to 0} x} = \frac{1-1}{0} = \frac{0}{0}$$

$$\text{OR} \quad \lim_{x\to\infty} \frac{e^{2x}}{x} - \frac{1}{x} = \infty - \infty$$

We can't figure this one out... YET!

When talking about limits we sometimes say things like

"x^2 goes to infinity faster than x"

$\hookrightarrow$ we can use rates = derivatives to help us evaluate these limits.

l'Hopitals Rule

If f and g are differentiable and for a real # a or for $\pm\infty$:

1. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

2. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided this limit exists.

Ex $\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{f(x)}{g(x)}$ Divided functions

AND $\lim_{x\to 0} e^{2x}-1 = 0$ $\lim_{x\to 0} x = 0$ $\frac{0}{0}$ case.

use l'Hopitals Rule:

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = \frac{2}{1} = \boxed{2}$$

Ex $\lim_{t \to 0} \frac{e^t - 1 - t}{t^2} = \frac{0}{0}$ both limits go to zero ... l'Hopital applies ...

$= \lim_{t \to 0} \frac{e^t - 1}{2t} = \frac{0}{0}$ NOW WHAT ? ... do it again ... l'Hopital still applies!

$= \lim_{t \to 0} \frac{e^t}{2} = \boxed{\frac{1}{2}}$

Ex $\lim_{x \to \infty} \frac{5x + e^{-x}}{7x} = \frac{\infty + 0}{\infty} = \frac{\infty}{\infty}$ both limits go to ∞ ... l'Hopital applies

$= \lim_{x \to \infty} \frac{5 + e^{-x}}{7} = \frac{5}{7}$

The growth of the function is helping us understand or solve the limit.

Ex $\lim_{x \to \infty} x e^{-x} = \infty \cdot 0$ problem ... but not in the correct form ...

Rewrite this as $\lim_{x \to \infty} \frac{x}{e^{+x}} = \frac{\infty}{\infty}$ NOW do l'Hopital.

$= \lim_{x \to \infty} \frac{1}{e^x} = 0$

Ex $\lim_{x \to \infty} \frac{x^2 + x}{x}$ ... well yes we could do l'Hopital <u>BUT</u>

DONT FORGET TO SIMPLIFY !!

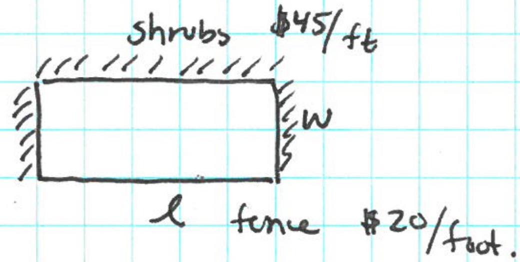
$= \lim_{x \to \infty} x + 1 = \infty$ DONE.

YOU TRY $\lim_{x \to 1} \frac{\ln(x)}{x^2-1}$ and $\lim_{x \to \infty} x^2 e^{-2x}$

More Application Problems...

<u>Ex</u> A landscaper plans to enclose a 3000 ft^2 rectangular region in a botanical garden. She will use shrubs costing \$45 per foot on three sides and fencing costing \$20 per foot on the fourth side. Find the minimum total cost.

DRAW



KNOW: Area $= \boxed{\ell w = 3000}$

WANT: Minimum Cost...

$$\text{Cost} = 45\ell + 45w + 45w + 20\ell$$
$$= 65\ell + 90w$$

$$\boxed{C = 65\ell + 90w}$$

PLUG IN... $\ell = \frac{3000}{w}$ so $C = \frac{65(3000)}{w} + 9w$

ONE UNKNOWN!

CALCULUS... MIN... $C' = -\frac{65(3000)}{w^2} + 9 = 0$

$$w^2 = \frac{65(3000)}{9} = 21666.67$$

$$w = \pm 18.257$$

TEST... $C'' = 2\left[\frac{65(3000)}{w^3}\right]$

for $w > 0$ $C'' > 0$ $\cup$ MIN!

Global Min? Endpoints ... $w=0$ $w \to \infty$

$C(0)$ = undefined .... divide by zero $\to \infty$

$\lim_{w \to \infty} \frac{65(3000)}{w^2} + 9$ = DNE + 9 undefined $\to \infty$

$w = 18.275$ is a global min.

Answer... Minimum total cost:

$$C(18.275) = \frac{65(3000)}{18.275} + 9(18.275) = \$10834.79.$$

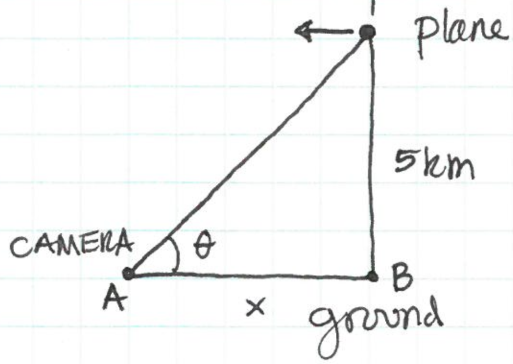
<u>EXTRA</u> What are the best possible bounds for TOTAL COST?

lower bound = global min = \$10834.79
upper bound = global max = undefined or unbounded.

Does this make sense, WHY?

Yes... there is a minimum she could spend but if the rectangle gets really long on one side... the cost goes really high!

Ex An airplane, flying at 450 km/hr at a constant altitude of 5 km is approaching a camera mounted on the ground. Let θ be the angle of elevation above the ground at which the camera is pointed. When $\theta = \pi/3$, how fast does the camera have to rotate to keep the plane in view.



we know that the plane is 5km above the ground.

let x be the distance along the ground between the plane and the camera.

Need something to represent θ:

$$\tan\theta = 5/x \quad \text{ok thats a start!}$$

How fast does the camera have to rotate?
looking for $\frac{d\theta}{dt}$ and $\theta = \theta(t)$ $x = x(t)$

lets take the derivative

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{-5}{x^2}\frac{dx}{dt}$$

need $\cos^2\frac{\pi}{3}$, x when $\theta = \pi/3$ and $\frac{dx}{dt}$ } unknowns.

now $\frac{dx}{dt} = 450$ km/hr

when $\theta = \pi/3$ $\quad x = 5/\tan(\pi/3) = \frac{5}{\sqrt{3}}$ $\quad \cos\frac{\pi}{3} = 1/2$

so: plug in and solve.

$$\frac{d\theta}{dt} = \frac{-5}{\left(\frac{5}{\sqrt{3}}\right)^2}\cdot\cos^2\frac{\pi}{3}\cdot 450\text{ km/hr} = 67.5\text{ rad/hour}$$

so the camera must rotate at about 1 rad per minute to keep the plane in view.