Theorems ABOUT DEFINITE INTEGRALS

We already have the FUNDAMENTAL THEOREM ... but these properties are really useful

Properties of limits of Integration

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Properties of Sums and Constant Multiples

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

Area Between two Curves

- assume $f(x)$ lies above $g(x)$ for $a \le x \le b$

$$\text{Area between } f \text{ and } g = \int_a^b \underset{\text{TOP}}{f(x)} - \underset{\text{BOTTOM}}{g(x)}\,dx$$

Symmetry to Evaluate Integrals

if f is even $\quad \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad$ Like x^2 or x^4

if f is odd $\quad \int_{-a}^{a} f(x)\,dx = 0 \quad$ Like x or x^3

LOTS OF EXAMPLES ...

Using the fundamental Theorem to Compute Integrals.

Ex Compute $\int_1^3 2x\,dx$

first use your calculator $\int_1^3 2x\,dx = 8$

but what function has the derivative $F'(x) = 2x$?

well $F(x) = x^2$ works!

so $\underbrace{\int_a^b F'(x)\,dx = F(b) - F(a)}_{\text{fundamental theorem}} \Rightarrow \int_1^3 2x\,dx = x^2\Big|_1^3 = (3)^2 - (1)^2 = 9 - 1 = 8$

same answer. ✓

Examples ... using the properties ...

<u>EX</u> Say we know that $f(x)$ is odd and that $\int_{-2}^{5} f(x)\,dx = 8$

<u>find</u> $\int_{2}^{5} f(x)\,dx =$

1. Break up the first integral   Properties of the Limits.

$$\int_{-2}^{5} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx = 8$$ KNOW THIS.

if $f(x)$ is odd then the first integral $= 0$

so $\int_{2}^{5} f(x)\,dx = 8$ so the value 8 comes from the second part only.

<u>EX</u> <u>find</u> $\int_{2}^{5} f(x)\,dx$ assuming we know that $\int_{2}^{5} (3f(x)+4)\,dx = 18$

Break up the integral   SUMS and CONSTANT MULTIPLES ...

$$3\int_{2}^{5} f(x)\,dx + 4\int_{2}^{5} dx = 18$$

Use the fundamental theorem ...
what function has $F'(x) = 1$? $F(x) = x$
we can solve this one!

$$4\int_{2}^{5} dx = 4\left(x\Big|_{2}^{5}\right) = 4(5-2) = 4(3) = 12 \checkmark$$

SOLVE ...

$$3\int_{2}^{5} f(x)\,dx + 12 = 18 \quad \text{so} \quad 3\int_{2}^{5} f(x)\,dx = 6$$

plug in

or $\int_{2}^{5} f(x)\,dx = 2$ ✓

Ex. Find the area of the region under $y=e^x$ and above $y=1$ for $0 \le x \le 2$

Area between f and g $= \int_a^b f(x) - g(x)\,dx \qquad f(x) < g(x)$

So the area of our region $= \int_0^2 e^x - 1\,dx$ use the fundamental thm.

$$= \int_0^2 e^x dx - \int_0^2 1\,dx = e^x\Big|_0^2 - x\Big|_0^2 = [e^2 - e^0] - [2-0]$$

$$= e^2 - 1 - 2 = e^2 - 3$$

$F'(x) = e^x$ so $F(x) = e^x$

$G'(x) = 1$
$G(x) = x$

compare w/ calculator...

We can also use integration to find <u>averages</u>

- If I had a bunch of numbers how would I find the average? Add them up divide by how many there were.

$$\frac{f_1 + f_2 + \cdots f_n}{n}$$

- Now

For a function $f(t)$ if I wanted the average value I could do a similar thing...

Approximate using a few values

$$\frac{f(t_1) + f(t_2) + \cdots f(t_n)}{(b-a)/\Delta t} = \frac{1}{(b-a)}\left[f(t_1) + f(t_2) + \cdots f(t_n)\right]\Delta t$$

But the more values we pick the more accurate our average!

$$\text{Average} = \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(t_i)\Delta t$$

Reimann Sum!

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

SAY WE ARE GIVEN:

assume $f(x)$ is EVEN

$$\int_0^4 f(x)\,dx = 8 \qquad \int_0^4 g(x)\,dx = 1$$

assume $g(x)$ is ODD

$$\int_1^4 f(x)\,dx = 2 \qquad \int_0^4 g^2(x)\,dx = 5$$

FIND THE FOLLOWING:

1. $\int_0^4 f(x) + g(x)\,dx$ $\qquad \int_0^4 f(x)\,dx + \int_0^4 g(x)\,dx = 8+1 = 9$

2. $\int_0^1 f(x)\,dx$ $\qquad \int_0^1 f(x)\,dx = \int_0^4 f(x)\,dx - \int_1^4 f(x)\,dx = 8-2 = 6$

3. $\left[\int_0^4 g(x)\,dx\right]^2$ $= [1]^2 = 1$

4. $\int_{-4}^4 f(x)\,dx$ EVEN $= 2\int_0^4 f(x)\,dx = 2(8) = 16$

5. $\int_{-4}^4 f(x) + g^2(x)\,dx$ $\qquad \int_{-4}^4 f(x)\,dx + \int_{-4}^4 g^2(x)\,dx = 2(8) + 2(5)$
   EVEN $\qquad$ square makes EVEN
   $= 16 + 10 = 26$

6. The Average value of $f(x)$ between $1 \le x \le 4$ $\qquad \frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{1}{3}(2) = \frac{2}{3}$

7. If $f(x) = F'(x)$ and $F(4) = 12$ what is $F(0)$?
   $\int_0^4 f(x)\,dx = F(4) - F(0)$ so $8 = 12 - F(0)$ so $F(0) = 4$

8. If $g(x)$ represents \$ per mile a driver is paid for driving x measured in 100's of miles what does $\int_0^4 g(x)\,dx$ represent?

   - The amt made in driving the first 400 miles.

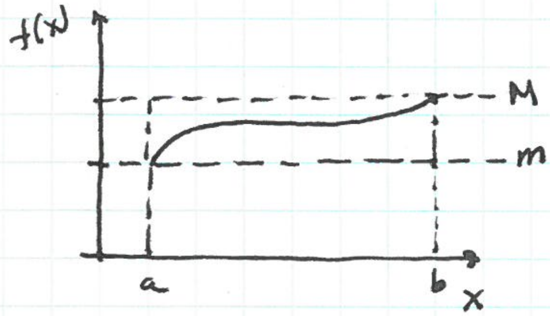
Comparing integrals

★ if $m \le f(x) \le M$ for $a \le x \le b$

then we say $f(x)$ is bounded below by m and bounded above by M

we also can say $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

so we have bounds for our integral



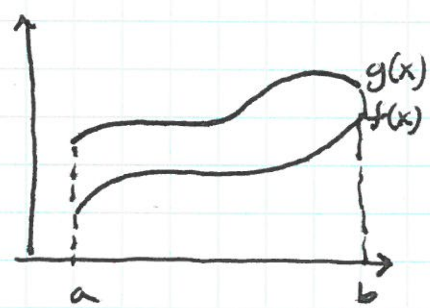
$M(b-a)$ is the BIG Rectangle

$m(b-a)$ is the LITTLE Rectangle

the area under the curve is definitely inbetween these.

★ if $f(x) \le g(x)$ for $a \le x \le b$ Then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



we can see the difference in the area under the curve.

Practice Problems:

(1) If $f(t)$ is measured in dollars per year and t is measured in years, what are the units of

$$\int_a^b f(t)\,dt \qquad \frac{\text{dollars}}{\text{year}} \cdot \text{years} = \text{dollars}.$$

(4) Explain in words what the integral means

$$\int_0^6 a(t)\,dt$$ where $a(t)$ is acceleration in km/hr^2 and t is time in hours.

First look at units: $a(t)dt = \frac{km}{hr^2} \cdot hr = \frac{km}{hr}$

so the integral has units of velocity

This is the change in velocity between $t=0$ and $t=6$, or over 6 hours.

(9) Find the average value of the function

$g(t) = 1+t$ over $[0,2]$

$$\text{Average} = \frac{1}{2-0}\int_0^2 1+t\,dt = \frac{1}{2}\left[t+\tfrac{1}{2}t^2\right]\Big|_0^2 = 2.$$

on your calculator or computer