2.3 Derivative function ...

In general we have been writing

$$f'(x) = \text{rate of change} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

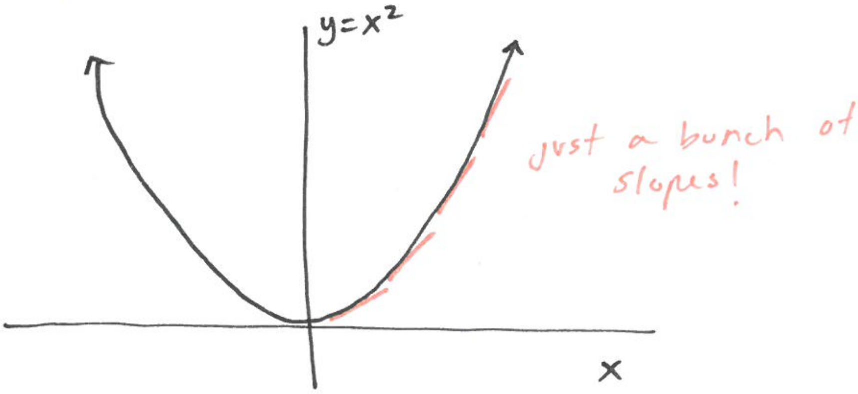
and then plugging in $x = a$ → a point.

Ex $y = x^2$ find the derivative at the following points...

$x = 1$
$x = 2$ and maybe some points inbetween.
$x = 3$
$x = 4$
⋮

hmm... this seems like a lot of work! Let's use our math brains to make this easier.

Since $y = x^2$ is a nice smooth function wouldn't the derivative EVERYWHERE be defined?



OBSERVATION

If $f(x)$ is increasing then $f'(x)$ is Positive
If $f(x)$ is decreasing then $f'(x)$ is Negative.

Lets try to find a function or formula for this derivative!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

only difference... we don't plug in a point !!

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \underbrace{h + 2x}_{} = 2x$$

Sometimes the book calls this <u>Difference Quotient</u>

so $f'(x) = 2x$.

<u>NOW</u> what is $f'(1) =$
$f'(2) =$
$f'(3) =$
$\vdots$

WAY BETTER !!

does this make sense on the graph..
$f'(-1)$
$f'(0)$

Let's find some more of these functions.

If $f(x) = mx + b$ find $f'(x)$ .... what is your intuition on this one?

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(m(x+h)+b) - (mx+b)}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = m$$ yup the slope!

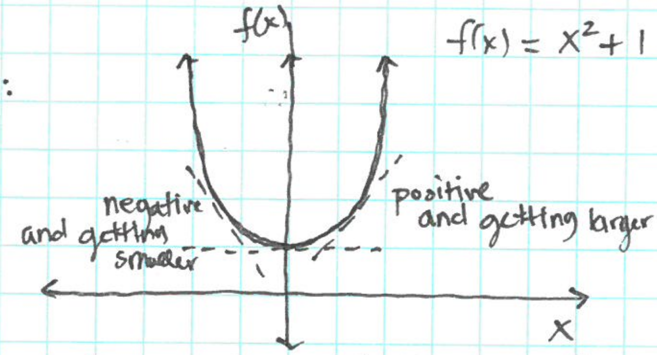
so $\frac{d}{dx}[mx+b] = m$

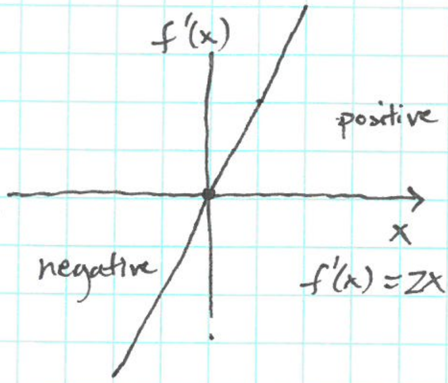
<u>YOU TRY</u> $f(x) = k$ a constant -and- $f(x) = x^3$

$$\boxed{\text{POWER RULE} \quad f(x) = x^n \text{ then } f'(x) = nx^{n-1}}$$

Memorize This!

Graphically:





Draw tangent lines and describe the slope in words!

<u>NOW</u>: If we can take the derivative of a function and get another function ... $f'(x) = g(x)$ then can't we do it again?

2.5 The Second Derivative.

NOTATION $$f''(x) = \left(f'(x)\right)' = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$$

these all denote the second derivative.

- all we are doing is taking the derivative again.

RECALL: If $f' > 0$ then f is increasing
If $f' < 0$ then f is decreasing

This means that:

A. If $f'' > 0$ then f' is increasing
B. If $f'' < 0$ then f' is decreasing

A. f' is increasing so the slope of $f(x)$ is getting larger and larger

$f'' > 0$ means concave up

B. f' is decreasing so the slope is getting smaller and smaller

$f'' < 0$ means concave down

By studying derivatives we can tell a lot about the shape of a function... but what does the second derivative mean?

when I see ... $\frac{d}{dx}$ I think ... "rate of change with respect to x"

so $\frac{d}{dx}\left(\frac{df}{dx}\right)$ means "rate of change wrt x of" "The rate of chang of f wrt. x"

The second derivative talks about how the change is changing

VELOCITY $= \frac{ds}{dt} = s'(t) = v(t)$ how quickly is the position changing

ACCELERATION $= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = s''(t)$ how quickly is the velocity changing

$= v'(t)$

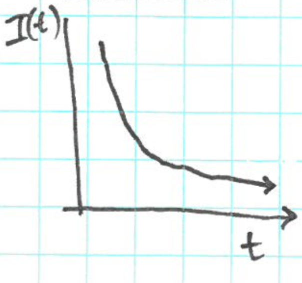
EX "The stock market has ~~been~~ ~~decreasing~~ dropping over the last few weeks ~~the~~ but the fall in prices seems to be slowing down. We hope that the stock market levels off so we can start our economic recovery"

$I(t)$ = stock market index at time t.

$I'(t) \longrightarrow$ Positive or Negative?
Negative because the prices are falling or decreasing.

$I''(t) \longrightarrow$ Positive or Negative?
Positive because the decrease is slowing down or the negative slope is becoming less severe

PICTURE



Quickly ... some definitions ...

DIFFERENTIABLE ... we can't always take the derivative ... why not ?

The function f is differentiable at x if

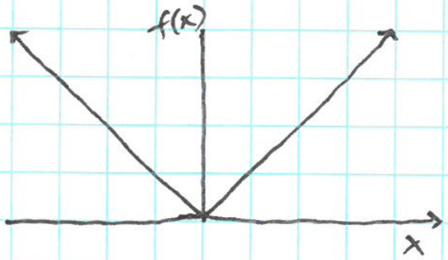
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists}$$

When does this happen ...

- when the graph has a vertical tangent line
- when the graph is not continuous or has breaks.
- when the graph has a sharp corner

<u>EX</u> $f(x) = |x|$ has a sharp corner at $x = 0$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$



At $x=0$ this becomes

$$\lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

The left and right hand limits don't match up.

From the left ... $\lim_{h \to 0^-} \frac{-h}{h} = -1$

From the right ... $\lim_{h \to 0^+} \frac{h}{h} = 1$

so the limit does not exist.

Think about this ... normally we zoom into a point and the curve looks like a tangent lin

zooming into this corner ... we will always get a corner.

The Relationship Between Differentiability and Continuity

Theorem 2.1: A Differentiable Function Is Continuous
If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.

Proof:

We assume f is differentiable at $x = a$. Then we know that $f'(a)$ exists. To show that f is continuous at $x = a$, we want to show that the limit as $x \to a$ of $f(x)$ is $f(a)$. We calculate the limit as $x \to a$ of $(f(x) - f(a))$, hoping to get 0. By algebra, we know that for $x \neq a$, $(f(x) - f(a)) = (x-a)\,(f(x) - f(a)) / (x-a)$. So

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left[(x - a) \frac{f(x) - f(a)}{x - a} \right]$$

$$= \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= 0 \cdot f'(a) = 0$$

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