- Today we are shifting gears and talking about how to calculate derivatives using shortcuts.

It is still Importand to be able to interpret what we are doing!

First The rules for differentiating a Polynomial

1. Taking the derivative of a Constant Multiple

$$\frac{d}{dx}\left[cf(x)\right] = c\,\frac{df}{dx} = cf'(x)$$

the constant just pops out frount

Ex $y = 3x$

line w/ slope of 3

$$\frac{dy}{dx} = \frac{d}{dx}(3x) = 3\,\frac{d}{dx}(x) = 3(1) = 3$$

2. Derivative of Sums or Differences

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$$

3. The Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Now lets use these rules and do a bunch of examples!

Ex. Find the derivative of $f(x) = 3x^2 + x^3$

$\frac{d}{dx}(f(x))$ 1. Simplify $f(x)$ and make everything into powers of x...

$f(x) = 3x^2 + x^3$ as simple as it gets

2. Take the derivative using the rules

$$\frac{df}{dx} = \frac{d}{dx}(3x^2) + \frac{d}{dx}(x^3)$$ sum and difference rule

$$= 3\frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$$ constant multiple rule

$$= 3(2x) + (3x^2)$$ power rule

3. Simplify $\frac{df}{dx} = 6x + 3x^2$

What if I asked you to find the equation of the tangent line at $f(1) = 4$

1. Equation of a line $y = mx + b$
2. Find the slope ... well we know the derivative at $(1,4)$

$\frac{df}{dx} = 6x + 3x^2$ at $(1,4)$ $\frac{df}{dx} = 9$

$m = 9$

3. $y = 9x + b$ now we know the line goes through $(1,4)$ so plug in and find b

$4 = 9 + b$ so $b = -5$

4. The eqn for the tangent line: $y = 9x - 5$.

Ex. Find the derivative of $f(x) = 2\sqrt{x} + \frac{1}{x} + x^2$

1. Simplify and write as powers of x

$$f(x) = 2x^{1/2} + x^{-1} + x^2$$

2. Take the derivative using the rule

$$\frac{df}{dx} = \frac{d}{dx}\left(2x^{1/2}\right) + \frac{d}{dx}\left(x^{-1}\right) + \frac{d}{dx}\left(x^2\right)$$ sum and difference

$$= 2\frac{d}{dx}\left(x^{1/2}\right) + \frac{d}{dx}\left(x^{-1}\right) + \frac{d}{dx}\left(x^2\right)$$ constant multiple

$$= 2\cdot\left(\tfrac{1}{2}x^{-1/2}\right) + \left(-1x^{-2}\right) + \left(2x\right)$$ power rule

3. Simplify

$$\frac{df}{dx} = x^{-1/2} - x^{-2} + 2x = \frac{1}{\sqrt{x}} - \frac{1}{x^2} + 2x$$

Find the equation of a tangent line at $f(1) = 4$

could choose any point this makes for easy algebra!

1. Eqn of a line $y = mx + b$

2. Find the slope $= m$

$$m = \left.\frac{df}{dx}\right|_{x=1} = \frac{1}{\sqrt{1}} - \frac{1}{(1)^2} + 2(1) = 2$$

3. $y = 2x + b$ we know it passes throug $(1,4)$ so plug in and solve for b

$$4 = 2(1) + b \qquad 4 = 2 + b \qquad b = 2$$

4. The eqn for the tangent: $y = 2x + 2$

Example $f(x) = (x+3)^2 - x$

Find the first and second derivative.

1. Find the first derivative
   (a) simplify and write in powers of x:

$$f(x) = x^2 + 6x + 9 - x \quad = x^2 + 5x + 9$$

   (b) Take the derivative using the rules

$$\frac{df}{dx} = \frac{d}{dx}(x^2) + 5\frac{d}{dx}(x) + \underbrace{\frac{d}{dx}(9)}_{\text{derivative of a constant}} \qquad \text{sum + dif constant}$$

$$= 2x + 5 \qquad \text{Power rule}$$

   (c) Simplify $\frac{df}{dx} = 2x + 5$

2. Now find the second derivative (use the same method)

$$\frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{d^2f}{dx^2} = \frac{d}{dx}(2x) + \frac{d}{dx}(5)$$

$$\frac{d^2f}{dx^2} = 2$$

Some Questions: Where is $f(x)$ increasing? $\Rightarrow$ slope positive $f'(x) > 0$

decreasing? $\Rightarrow$ slope negative $f'(x) < 0$

Concave up? $f'' > 0$ | Concave down? $f'' < 0$

$$\frac{df}{dx} = 0 = 2x + 5 \qquad x = -\frac{5}{2}$$

so for $x > -\frac{5}{2}$ $f(x)$ is increasing

$x < -\frac{5}{2}$ $f(x)$ is decreasing

Always concave up.

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Exercises on Power and Combining Rules

Find formulas for the derivatives of the functions in problems 1-4.

1. $f(x) = \sqrt[4]{x} + \frac{2}{x^4}$ $= x^{1/4} + 2x^{-4}$ so $f'(x) = \frac{1}{4}x^{-3/4} - 8x^{-5}$

2. $f(x) = 8x^4 + 5^7 - 3x^{2/3}$ so $f'(x) = 32x^3 + \emptyset - 2x^{-1/3}$

   hey sneaky! This is a constant

3. $f(x) = \frac{x - 1 + x^3}{x^2}$ (Hint: first use algebra to rewrite this in another form.)

   $f(x) = x^{-1} - x^{-2} + x$ so $f'(x) = -x^{-2} + 2x^{-3} + 1$

4. $f(x) = (3x^2 - 1)^2$ (Hint: again, use algebra first.)

   $f(x) = 9x^4 - 6x^2 + 1$ so $f'(x) = 36x^3 - 12x + \emptyset$

5. **Patterns of derivatives:** In each of the following cases, take the first derivative, the second derivative, the third derivative (i.e., the derivative of the second derivative, and the fourth derivative.

   (a) $f(x) = x^3 + 2x^2 - 14x + 33$

   $f'(x) = 3x^2 + 4x - 14$

   $f''(x) = 6x + 4$

   $f'''(x) = 6$ $f^{(4)}(x) = 0$

   (b) $f(x) = x^3 - 22x^2 - 72$

   $f'(x) = 3x^2 - 44x$ $f^{(3)}(x) = 6$

   $f''(x) = 6x - 44$ $f^{(4)}(x) = 0$

   (c) $f(x) = x^3 + 387x + 90$

   etc...

   (d) $f(x) = x^3 + 47x^2 - 16x$

6. Look at your answers to problem 5. What do you see that they have in common? Can you explain why? They all have the same fourth derivative! The other derivatives are just multiples of the power!

7. **Bonus problem!** If $f(x) = x^4 + 13x^3 - 55x^2 + 8x - 129$, find the fourth and fifth derivatives of $f(x)$ **without** actually taking any derivatives. (Wow! Can this really be done?)

   $f^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0 + 13 \cdot 3 \cdot 2 \cdot 1 \cdot 0$ all others would be zero...

   $= 24$

   $f^{(5)}(x) = 0$