Lecture #9 Calc 1

RULES FOR DIFFERENTIATION

1. $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$
2. $\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x) + g'(x)$
3. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
4. $\frac{d}{dx} e^x = e^x$
5. $\frac{d}{dx} a^x = \ln(a)a^x$

TODAY: how do we take the derivative of a product of two functions.

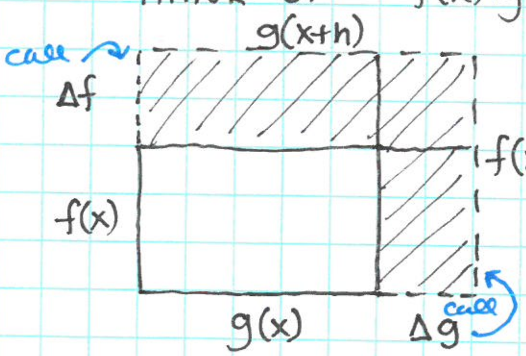
Definition of the derivative...

How do we find → $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

ok lets see if we can understand this limit...

Rewrite it Visually or Geometrically...

THINK of $f(x)\cdot g(x)$ as the area of a rectangle



Then $f(x+h)$ extends $f(x)$
$g(x+h)$ extends $g(x)$

and $f(x+h)g(x+h)$ is a larger rectangle

lets call Δf the amt that f changes: $f(x+h) - f(x)$
and Δg the amt that g changes: $g(x+h) - g(x)$

THEN: $f(x+h)g(x+h) - f(x)g(x)$ is the big rectangle minus the little rectangle OR the area of increase.

FROM OUR PICTURE: What is this area?

$$\Delta f\, g(x) + \Delta g\, f(x) + \Delta f\, \Delta g$$

so we rewrite our limit...

plug in...

$$= \lim_{h\to 0} \frac{\Delta f\, g(x) + \Delta g\, f(x) + \Delta f\, \Delta g}{h}$$

separate denominator

$$= \lim_{h\to 0} \frac{\Delta f\, g(x)}{h} + \lim_{h\to 0} \frac{\Delta g\, f(x)}{h} + \lim_{h\to 0} \frac{\Delta f\, \Delta g}{h}$$

but $\Delta f =$ $\qquad$ $\Delta g =$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x) + \lim_{h\to 0} \frac{\Delta f\, \Delta g}{h}$$

$$= \frac{df}{dx} g(x) ✓ + \frac{dg}{dx} f(x) ✓ + \lim_{h\to 0} \cancel{\frac{\Delta f}{h} \cdot \frac{\Delta g}{h}} \cdot h \to 0 \Rightarrow \frac{df}{dx} \cdot \frac{dg}{dx} \lim_{h\to 0} h.$$

mult. by a fancy 1. $\frac{h}{h} = 1$

BC Δf and Δg change as $h \to 0$

so we found:

$$\boxed{\frac{d}{dx} f(x) g(x) = f'(x) g(x) + g'(x) f(x)}$$

THE PRODUCT RULE!

Helps to remember the words! "derivative of the first times the second plus derivative of the second times the first."

<u>EX</u> $f(x) = x^2 e^x$

$$\frac{df}{dx} = \frac{d}{dx} x^2 e^x \overset{\text{product rule}}{=} \frac{d}{dx}(x^2) \cdot e^x + x^2 \frac{d}{dx} e^x = \overset{\text{power rule}}{2xe^x} + \overset{\text{exponential}}{x^2 e^x}$$

$$= (2x + x^2) e^x \quad \text{factor out.}$$

NOW – how do we deal with two functions that are divided?

Quotient

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \qquad \text{sneaky trick let } Q(x) = \frac{f(x)}{g(x)}$$

then $f(x) = Q(x) g(x)$

eventually we want $\frac{dQ}{dx}$ so lets take the derivative

$$\frac{df}{dx} = \frac{dQ}{dx} g + Q \frac{dg}{dx} = \frac{dQ}{dx} \cdot g(x) + \frac{f(x)}{g(x)} \frac{dg}{dx}$$

product rule

OR $$f'(x) = \frac{dQ}{dx} g(x) + \frac{f(x)}{g(x)} g'(x)$$

mult. through by $g(x)$ to get that $g(x)$ out of denominator

$$f'(x) g(x) = \frac{dQ}{dx} g^2(x) + f(x) g'(x)$$

we want derivative of the Quotient.

now solve for $\frac{dQ}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$

$$\frac{dQ}{dx} g^2(x) = f'(x) g(x) - f(x) g'(x)$$

$$\boxed{\frac{dQ}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}}$$

THIS IS THE QUOTIENT RULE.

Helps to remember the words!

"derivative of top times the bottom minus derivative of the bottom times the top all over the bottom squared"

Ex $f(x) = \frac{e^x}{x^2}$

$$\frac{df}{dx} = \frac{\frac{d}{dx}(e^x) \cdot x^2 - e^x \frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2 e^x - 2x e^x}{x^4} = \frac{(x^2 - 2x) e^x}{x^4}$$

$$= \frac{(\cancel{x} - 2) e^x}{x^3}$$

NOTE ... we could also do product rule here ...

$f(x) = e^x \cdot x^{-2}$ then $\frac{df}{dx} = \frac{d}{dx}(e^x) \cdot x^{-2} + e^x \frac{d}{dx} x^{-2}$

$$= \frac{e^x}{x^2} - \frac{2e^x}{x^3}$$

is this the same? $\Rightarrow \frac{(x-2)e^x}{x^3}$

YES

DO LOTS OF EXAMPLES.

EX $f(x) = (3x^2 + 5x)e^x$

1. SIMPLIFY? we could distribute the e^x... but it actually works better as it is.

2. Apply Product Rule:

$$\frac{df}{dx} = \underbrace{\frac{d}{dx}(3x^2+5x)}_{\text{power rule}} \cdot e^x + (3x^2+5x)\underbrace{\frac{d}{dx}e^x}_{\text{exponential}}$$

$$= (6x+5)e^x + (3x^2+5x)e^x$$

$$= (3x^2 + 11x + 5)e^x$$ Simplify for final answer.

EX $f(x) = \frac{1}{1+e^x}$ we can use the quotient rule here

TOP = 1 BOTTOM = $1+e^x$

1. simplify – it is already pretty simple.

2. Apply Quotient Rule

$$\frac{df}{dx} = \frac{\frac{d}{dx}(1)\,(1+e^x) - 1\cdot\frac{d}{dx}(1+e^x)}{(1+e^x)^2}$$

derivative of constant sum/difference exponent.

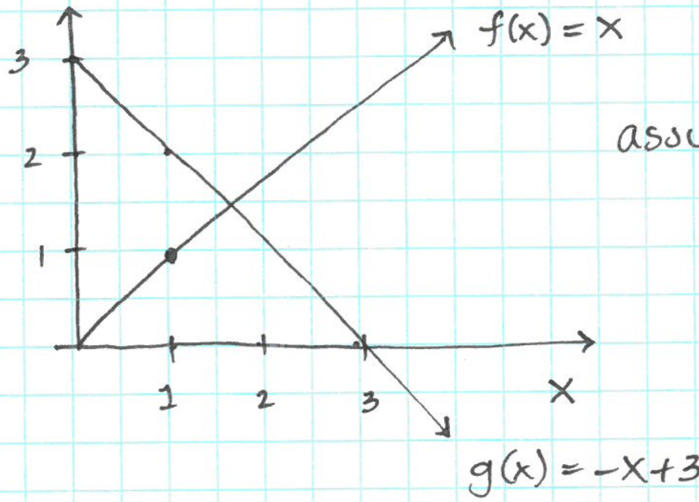
$$= \frac{0\cdot(1+e^x) - 1\cdot e^x}{(1+e^x)^2}$$

$$= \frac{-e^x}{(1+e^x)^2}$$ This is probably simple enough... we could foil the denom.

Ex

How can we see this from a graph?

no need to visualize what $f \cdot g$ looks like!



assume $h(x) = f(x)g(x)$

use the graph to find $h'(1)$

use the product rule $\dfrac{dh}{dx} = \dfrac{df}{dx} \cdot g + f \dfrac{dg}{dx}$

Translate the math into your own WORDS!

what does this mean in words...

$= \text{slope of } f \cdot \text{value of } g + \text{value of } f \cdot \text{slope of } g$

so $h'(1) = f'(1)\,g(1) + f(1)\,g'(1)$

$= \underset{\text{SLOPE}}{(1)}\underset{\text{VALUE}}{(2)} + \underset{\text{VALUE}}{(1)}\underset{\text{SLOPE}}{(-1)}$

$= 2 - 1 = 1$

so $h'(1) = 1$ ✓

Check this $h(x) = x(-x+3) = -x^2 + 3x$

$h'(x) = -2x + 3$ so $h'(1) = -2(1) + 3 = 1$

IT WORKS!!