Problem 3 (25 points)

Calculate the derivative of the following functions using the shortcuts learned in class:

$f(x) = x^2 + 3^x - e^x + 4x^2 - 1$

$\boxed{f'(x) = 2x + \ln(3)3^x - e^x + 8x}$ general rules.

$f(y) = \dfrac{y^2 - y + 1}{x}$ Typo... this should be a y. then $f(y) = y - 1 + 1/y$

$\boxed{f'(y) = 1 - \frac{1}{y^2}}$

derivative with respect to y and x is constant $f'(y) = \dfrac{2y-1}{x}$

$f(x) = (x^3 + x - 9)e^x$ product rule.

$\boxed{f'(x) = (3x^2+1)e^x + (x^3+x-9)e^x}$

$f(x) = (x^3 + x - 9)^{20}$ chain rule.

$\boxed{f'(x) = 20(x^3+x-9)^{19} \cdot (3x^2+1)}$

$g(\theta) = \sin(\sqrt{\theta})$ chain rule.

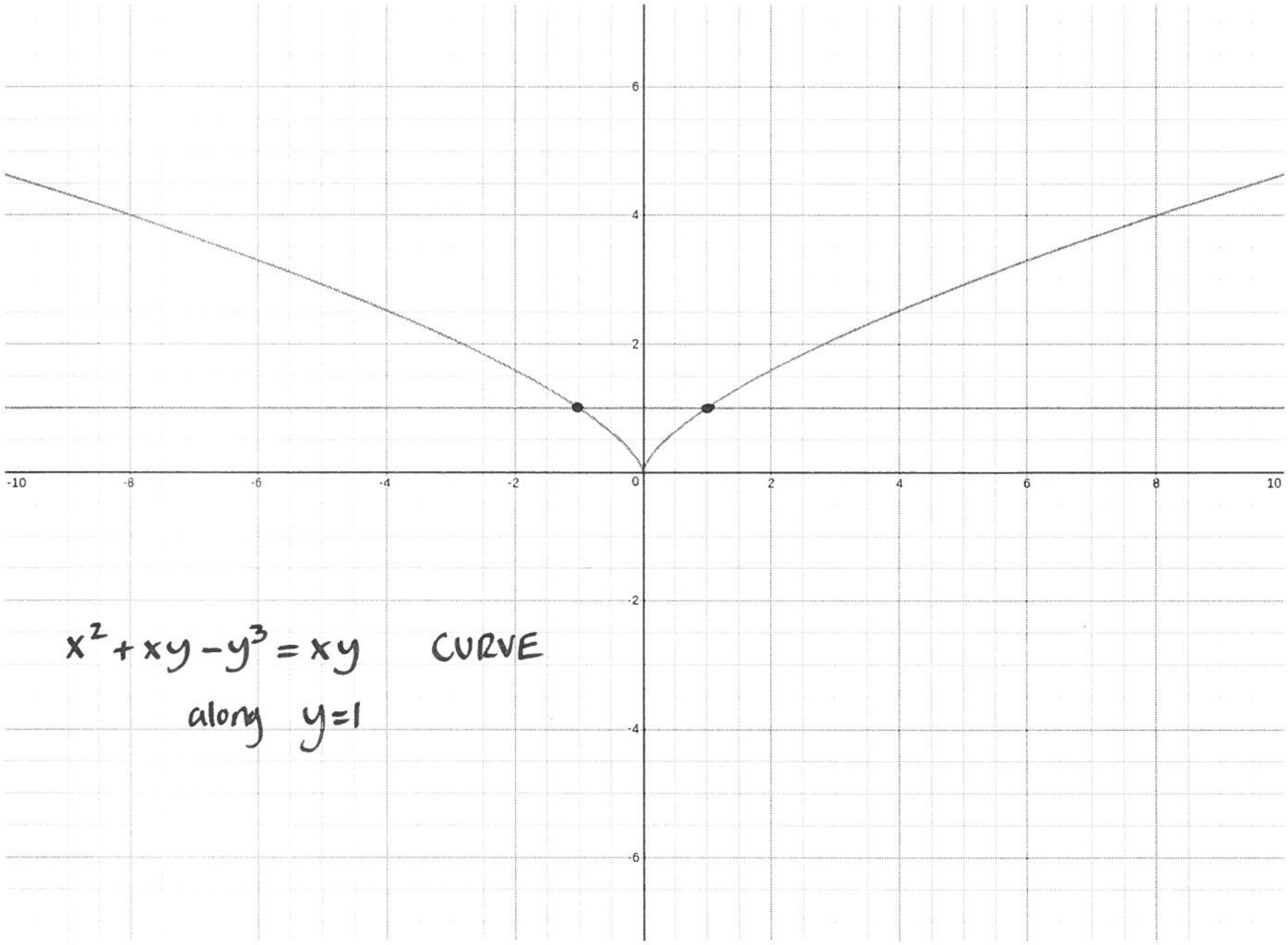
$\boxed{g'(\theta) = \cos(\sqrt{\theta}) \cdot \frac{1}{2\sqrt{\theta}}}$

$s(t) = e^{\tan(t)}$ chain rule

$\boxed{s'(t) = e^{\tan(t)} \cdot \frac{1}{\cos^2(t)}}$

$f(x) = \arctan(x^3 + e^x)$ chain rule.

$\boxed{f'(x) = \frac{1}{1+(x^3+e^x)^2} \cdot (3x^2+e^x)}$



$x^2+xy-y^3=xy$ CURVE

along $y=1$

All points on CURVE where $y=1$ plug $y=1$ into the curve.

$$x^2+\cancel{x(1)}-(1)^3=\cancel{x(1)} \qquad x^2=1 \quad \text{then} \quad x=\pm 1$$

points $(-1,1)$ and $(1,1)$

Expect negative slope at $(-1,1)$
positive slope at $(1,1)$

DERIVATIVE ... first $x^2+xy-y^3=xy$ can be simplified!

$$x^2-y^3=0$$

then

$$\frac{d}{dx}\left[x^2-f(x)^3\right]=\frac{d}{dx}\left[0\right]$$

$$2x-3f(x)^2f'(x)=0$$

$$f'(x)=\frac{2x}{3f(x)}=\frac{2x}{3y}$$

Tangent Lines – there are two different points.

For $(-1,1)$ $\quad$ slope $= y'\big|_{(-1,1)} = \frac{2(-1)}{3(1)} = -\frac{2}{3} = m$

$y = -\frac{2}{3}x + b$ $\quad$ plug in $(-1,1)$ to find b

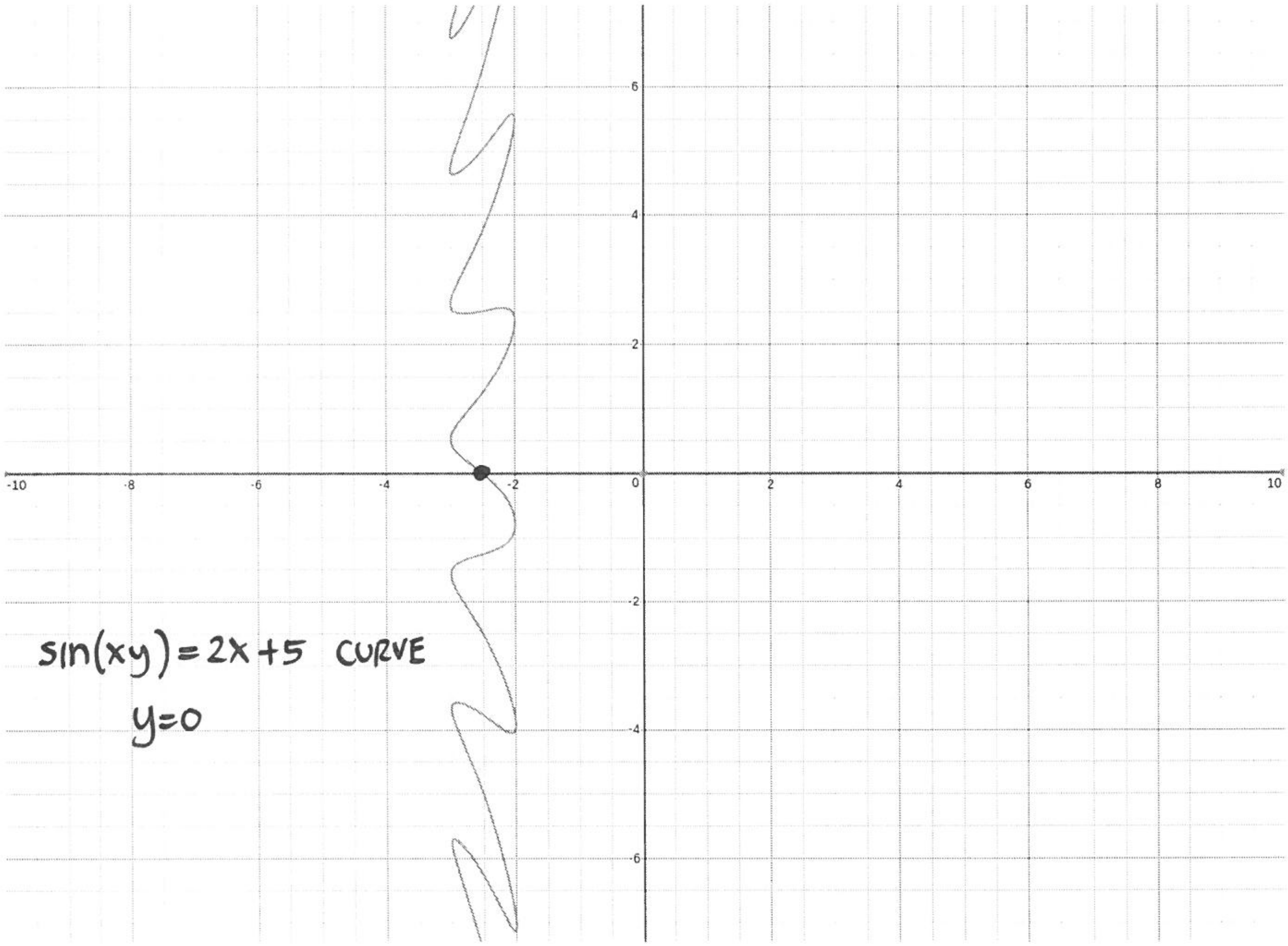
$1 = -\frac{2}{3}(-1) + b$ $\quad$ so $b = \frac{1}{3}$

$y = -\frac{2}{3}x + \frac{1}{3}$

For $(1,1)$ $\quad$ slope $= y'\big|_{(1,1)} = \frac{2(1)}{3(1)} = \frac{2}{3} = m$

$y = \frac{2}{3}x + b$ $\quad$ plug in $(1,1)$ to find b

$1 = \frac{2}{3}(1) + b$ $\quad$ so $\quad b = \frac{1}{3}$ $\quad \boxed{y = \frac{2}{3}x + \frac{1}{3}}$



Plug in $y=0$ to find x ... $\sin(x(0)) = 2x+5$

$0 = 2x+5$ $\quad x = -5/2$

matches graph.

Point $(-5/2, 0)$ Expect negative slope.

Derivative: $\frac{d}{dx}\left[\sin(x \cdot f(x))\right] = \frac{d}{dx}\left[2x+5\right]$

chain <u>and</u> product.

$$\cos(x \cdot f(x)) \cdot \left[f(x) + x f'(x)\right] = 2$$

$$\cos(xy)\, y + \cos(xy)\, x\, y' = 2$$

$$y' = \frac{2 - \cos(xy) y}{x \cos(xy)}$$

Tangent line at $(-5/2, 0)$

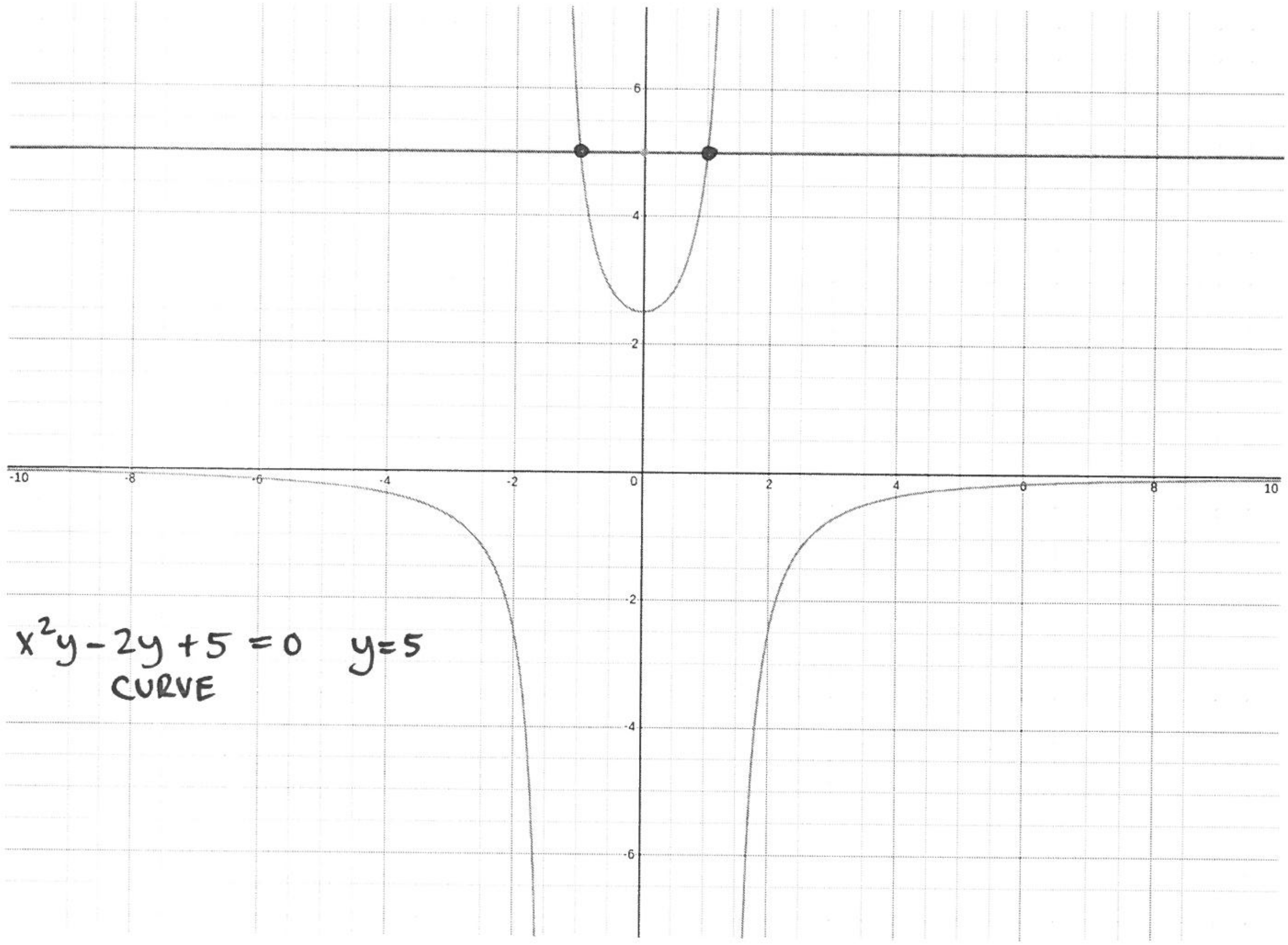
$$\text{slope} = y'\Big|_{(-5/2,\,0)} = \frac{2 - \cos\left(-\frac{5}{2}\cdot 0\right)\cdot 0}{-5/2\,\cos\left(-\frac{5}{2}\cdot 0\right)} = \frac{2}{-5/2} = -4/5 = m$$

$$\cos(0) = 1$$

$y = -4/5\,x + b$ plug in $(-5/2, 0)$ to find b.

$$0 = (-4/5)(-5/2) + b = 2 + b \qquad b = -2$$

$$\boxed{y = -4/5\,x - 2}$$



Points on CURVE where $y=5$: $x^2(5) - 2(5) + 5 = 0$

$5x^2 - 5 = 0$

$x^2 = 1 \quad x = \pm 1$

two points $(-1,5)$ and $(1,5)$

Expect Neg. Slope — Expect pos Slope.

Derivative: $\frac{d}{dx}\left(x^2y - 2y + 5\right) = \frac{d}{dx}(0)$

NOTE $y = f(x)$

product.

$2xy + x^2y' - 2y' = 0$

$$y' = \frac{-2xy}{x^2 - 2}$$

Tangent line at $(-1, 5)$

$$m = y'\Big|_{(-1,5)} = \frac{-2(-1)(5)}{(-1)^2 - 2} = \frac{10}{-1} = -10$$

$y = -10x + b$ $\qquad 5 = -10(-1) + b \qquad b = -5$

$$\boxed{y = -10x - 5}$$

Tangent line at $(1, 5)$

$$m = y'\Big|_{(-1,5)} = \frac{-2(1)(5)}{(1)^2 - 2} = \frac{-10}{-1} = 10$$

$y = 10x + b \qquad 5 = 10(1) + b \quad b = -5$

$$\boxed{y = 10x - 5}$$

The tangent line is horizontal when

$$y' = \frac{-2xy}{x^2 - 2} = 0$$ this only happens if the NUMERATOR $= 0$

$-2xy = 0$ when $x = 0$ or $y = 0$
check on the CURVE
... aka plug into $x^2y - 2y + 5 = 0$

$x = 0 \Rightarrow 0^2 y - 2y + 5 = 0$
$y = 5/2$ $\qquad \boxed{(0, 5/2) \quad \text{SLOPE} = 0}$

$y = 0 \Rightarrow x^2(0) - 2(0) + 5 = 0$ or $5 = 0$ NO

The tangent line is undefined when

$$y' = \frac{-2xy}{x^2 - 2} = \text{undefined}$$ or when the DENOMINATOR $= 0$

$x^2 - 2 = 0$ when $x = \pm\sqrt{2}$

plug in $(\sqrt{2})^2 y - 2y + 5 = 0 \qquad 5 = 0$ No points on curve
BUT... be can see on the graph
this is an asymptote!

Problem 5 (10 points)

Using the functions plotted on the next page, say if the following are positive, negative, zero, or undefined. You MUST explain how you came to your conclusion.

(a) $f(3)$ positive ... on the graph when $x=3$
$f(3) \sim 5$.

(b) $f'(3)$ negative ... the slope when $x=3$
is negative.

(c) $f''(3)$ negative ... the graph is concave down
at $x=3$.

(d) $\frac{d}{dx}(f(3)g(3))$ $= \underset{\text{NEG}}{f'(3)}\,\underset{\text{POS}}{g(3)} + \underset{\text{NEG}}{g'(3)}\,\underset{\text{POS}}{f(3)}$ = NEGATIVE + NEGATIVE
product rule

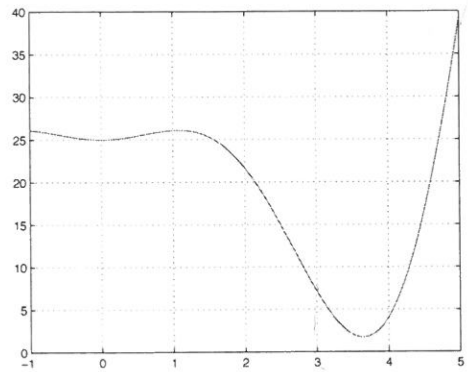
so $\frac{d}{dx}\left[f(3)\,g(3)\right]$ is negative.

(e) $\frac{d}{dx}(c \cdot g(3))$ where c is a positive constant.

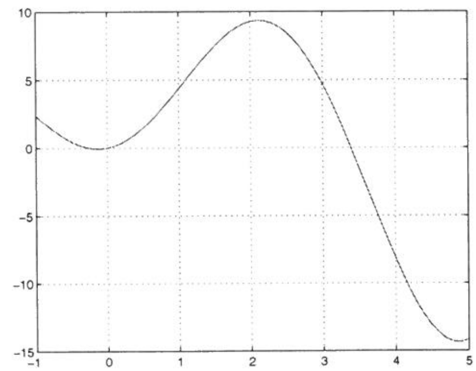
$= c\,g'(3) =$ pos $\cdot$ neg $=$ negative.

(f) $\frac{d}{dx}(g(f(3)))$ $= \underset{\text{POS}}{g'(f(3))} \cdot \underset{\text{NEG}}{f'(3)}$ = negative.
chain rule

$f(3)=5$ and $g'(5)$ is pos.



(a) A graph of $g(x)$



(b) A graph of $f(x)$

Problem 7 (20 points)

Consider the equation $f(x) = x^3 + 3x^2 + 4$.

SLOPE $f'(0)$

(a) Find the equation of the tangent line at $x = 0$.

$y = mx + b$

need a point $f(0)$

$f'(x) = 3x^2 + 6x$

$f'(0) = 0$ so $m = 0$

$f(0) = 4$ so $(0, 4)$ is a point on the line.

$y = \cancel{0}x + b$ or $y = b$ and $(0,4) \Rightarrow b = 4$

so $\boxed{y = 4}$

(b) Where is $f(x)$ concave up?

Concave need $f''(x) = 6x + 6$

concave up $f''(x) = 6x + 6 > 0$

$6x > -6$ $\boxed{x > -1}$

(c) Where is $f(x)$ is increasing?
HINT: There should be two conditions $x > a$ and $x < b$ where a and b are numbers that you find. Plug in numbers if you're not sure.

increasing $f'(x) = 3x^2 + 6x > 0$

$x(3x + 6) > 0$

$3x(x + 2) > 0$

either BOTH Positive OR BOTH Negative.

$3x > 0$ and $x + 2 > 0$

$x > 0$ and $x > -2$

SATISFIES

$3x < 0$ $x + 2 < 0$

$x < 0$ $x < -2$

satisfies.

$f(x)$ is increasing when $x < -2$ and when $x > 0$

Consider the function $f(x) = \sqrt{1+x}$, your goal is to find a linear approximation for $f(0.1)$. Use the following steps:

1. What "easy" point is near the point $x = .1$ where we would like to make our approximation? In other words, what x value near $x = 0.1$ would be WAY easier to plug into our function?
2. Find the tangent line approximation for $f(x)$ near the point $x = a$, where a is the "easy" point that you just found. You will need the following pieces:
   - $f(a) =$
   - $f'(a) =$
3. Put all of the pieces together to write out the tangent line approximation.
4. Now plug the point $x = 0.1$ into the tangent line approximation you just found. This gives you your answer "an approximate value for $f(0.1)$"
5. Use the Error formula to say what your approximate error is.
6. Now USING YOUR CALCULATOR FOR THE FIRST TIME.... find the "real" value for $f(0.1) = \sqrt{1.1}$. How close was your answer?

1. It is way easier to plug in $x=0$ $\quad f(0)=\sqrt{1}=1$
and $x=0$ is pretty close to $x=.1$

2. Find tangent line approx near $x=a=0$

$$f(0)=\sqrt{1+0}=\sqrt{1}=1$$

$$f'(x)=\frac{d}{dx}(1+x)^{1/2}=\frac{1}{2}(1+x)^{-1/2}=\frac{1}{2\sqrt{1+x}} \qquad f'(0)=\frac{1}{2\sqrt{1}}=1/2$$

3. $$f(x)=\sqrt{1+x}\sim 1+1/2[x-\cancel{0}]=1+\frac{x}{2}$$

4 $$f(0.1)\sim 1+\frac{.1}{2}=1+.5(.1)=1+.05=1.05$$

5 $$E(.1)\cong \frac{f''(0)}{2}[x-0] \quad \text{need} \quad f''(x)=\frac{d}{dx}\left[\frac{1}{2}(1+x)^{-1/2}\right]=-\frac{1}{4}(1+x)^{-3/2}$$

$$f''(0)=-\frac{1}{4}(1)^{-3/2}=-1/4$$

$$E(.1)\sim \frac{-1/4}{2}[.1]=-\frac{1}{8}\cdot[.1]=-.0125$$

6. TEST $f(.1)=\sqrt{1.1}=1.0488$

Use the ideas above to find an approximate value for $\sqrt{4.2}$ and talk about how close your approximation is. NO FAIR USING A CALCULATOR OR COMPUTER OR PHONE OR.....

Find an approximate value for $\sqrt{4.2}$.

Well $\sqrt{4} = 2$ is one I know! So $f(x) = \sqrt{x}$ near $x = 4$

linear approximation.

$$f(x) \sim f(a) + f'(a)[x - a] \qquad a = 4$$

$$f(4) = 2 \qquad f'(x) = \frac{1}{2\sqrt{x}} \quad f'(4) = \frac{1}{4}$$

$$\sqrt{x} \sim 2 + \tfrac{1}{4}[x - 4] \text{ near } x = 4$$

then $$f(4.2) \sim 2 + \tfrac{1}{4}[4.2 - 4] = 2 + \tfrac{1}{4}[.2] = 2 + .25[.2]$$

$$= 2.05$$

$$\lim_{x\to 0} \frac{\ln(1+7x)}{2x}$$

$$\left.\begin{array}{l} f(0)=\ln(1)=0 \\ g(0)=0 \\ g'(x)=2 \ \text{ so } \ g'(0)=2 \end{array}\right\} \text{Use L'Hopitae!}$$

$$\lim_{x\to 0} \frac{\ln(1+7x)}{2x} = \lim_{x\to 0} \frac{\frac{7}{1+7x}}{2} = \frac{7}{2}$$

$$\lim_{x\to \pi} \frac{\sin(x)}{x-\pi}$$

$$\left.\begin{array}{l} f(\pi)=\sin(\pi)=0 \\ g(\pi)=0 \\ g'(x)=1 \quad g'(0)=1 \end{array}\right\} \text{Use L'Hopitae}$$

$$\lim_{x\to \pi} \frac{\sin(x)}{x-\pi} = \lim_{x\to \pi} \frac{\cos(x)}{1} = -1$$

since $\cos(\pi)=-1$
we approach

$$\lim_{x\to 1} \frac{x^2-1}{1+\ln(x)}$$

$f(1)=0$
$g(1)=2$

don't need L'Hopitae!

$$\lim_{x\to 1} \frac{x^2-1}{1+\ln(x)} = \frac{\lim_{x\to 1} x^2-1}{\lim_{x\to 1} 1+\ln(x)} = \frac{0}{1} = 0$$

$$\lim_{x\to \infty} \frac{x^2}{\ln(x)}$$

$$\left.\begin{array}{l} \lim_{x\to \infty} x^2 \text{ approaches } \infty \\ \lim_{x\to \infty} \ln(x) \text{ approaches } \infty \end{array}\right\} \text{Use L'Hopitae.}$$

$$\lim_{x\to \infty} \frac{x^2}{\ln(x)} = \lim_{x\to \infty} \frac{2x}{\frac{1}{x}} = \lim_{x\to \infty} 2x^2 = \infty \text{ or DNE.}$$

Problem 9 (5 points)

Extra Challenge Problem - on most exams I include a challenge problem that is more difficult but not worth a ton of points. This problem is the difference between an A and and A+ on the exam. Here are some examples:

1. Prove the power rule for non integer values of n using inverses, the chain rule, and the power rule.

$f(x) = x^{1/m}$ where m is any intiger

then $[f(x)]^m = x$ and $m[f(x)]^{m-1} f'(x) = 1$ by the chain rule.

so $f'(x) = \frac{1}{m}[f(x)]^{-(m-1)} = \frac{1}{m} f(x)^{\frac{1}{m}-1}$ ✓

2. Find the derivative of

$$f(x) = \arccos(x)$$

using inverses. Simplify your answer using trigonometric rules.

$\cos(f(x)) = x$ then $\sin(f(x))f'(x) = 1$ so $f'(x) = \frac{1}{\sin(f(x))}$

but $\sin^2(x) = 1 - \cos^2(x)$ so $f'(x) = \frac{1}{\sqrt{1-\cos^2(f(x))}}$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ ✓

3. Show that you can get the quotient rule using the product rule.

let $Q(x) = \frac{f(x)}{g(x)}$ then $f(x) = Q(x)g(x)$

$f'(x) = Q'(x)g(x) + g'(x)Q(x)$

$f'(x) = Q'(x)g(x) + g'(x) \cdot \frac{f(x)}{g(x)}$ solve for $Q'(x)$

$Q'(x)g(x) = f'(x) - g'(x) \cdot \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$ ✓