MATH 121
Practice Problems for Midterm 1

Name: Solutions

	Score
1	/5
2	/5
3	/5
4	/5
5	/5
6	/5
7	/5
Total	/35

These practice problems are not intended to be an exhaustive list of problems for the exam.

Any problem type that we have done on the homework or in class is fair game for the exam. You should go watch the homework videos to see some great solutions!

The exam will consist of just 3-4 randomly assigned problems.

If you understand all of the concepts you see here then you should be prepared for the exam. If you memorize the problems here, you will not be prepared for the exam, because the problems will be different.

On the exam you should show as much work as possible to get as much partial credit as possible. I am very generous with partial credit when it is clear that students are making an effort to explain themselves.

Things you should know for this exam:
** All of the basic algebra rules for functions.
** How to take a limit of a function.
** What it means for a function to be continuous and/or differentiable.
** Definition of average and instantaneous velocity.
** The central difference approximation for estimating the derivative.
** How to find a derivative algebraically using the limit and how to find a derivative numerically using data.
** What the derivative means. This includes saying in words what $f(x)$, $f'(x)$ and $f''(x)$ means given some information about $f(x)$.
** How to identify the derivatives on a graph.
** Slope and Curvature as they are related to the first and second derivatives.
** Finding the equation of a tangent line.
** Using the derivative to approximate a function near a point. Eg. Given $f'(a)$ and $f(a)$ what is an approximate value of $f(a + .5)$

Problem 1 (5 points)

(a) Draw two functions: one that is continuous and one that is not continuous. Use the formal definition of continuity to describe why one is continuous everywhere and why the other is not.
(b) Draw a function that is continuous but NOT differentiable. Explain in words why the function is continuous but not differentiable.
(c) Draw a function that has the following qualities:

$$\lim_{x \to 0^+} f(x) = 0 \quad lim_{x\to 0^-} f(x) = 1 \quad f'(2) = -1$$

(d) Sketch the function $y = x^2 - 4$ on your sketch lines on the graph that represent the following items:

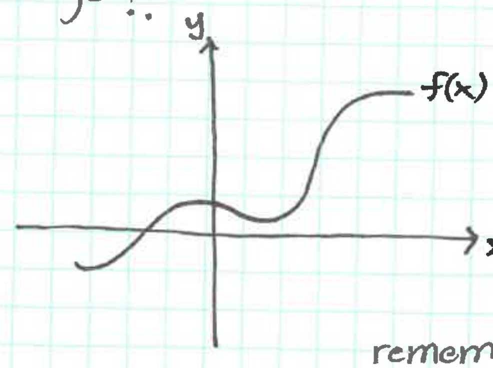
- $f'(0)$
- $f'(1)$
- The average ~~slope~~ rate of change. between $f(0)$ and $f(1)$

For each of the lines you drew explain WHY it represents the given item and give the exact value for the calculation.

Everyone will have different drawings !!

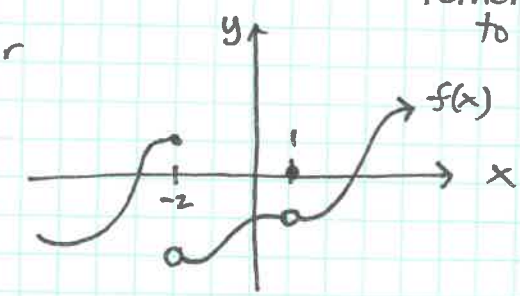
(a)

Continuous = can draw without picking up my pen.



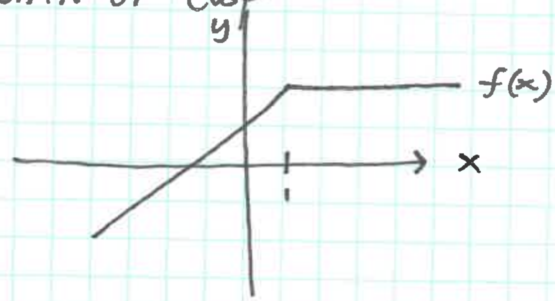
remember to label axes !

Not continuous = has jumps or holes in the graph.



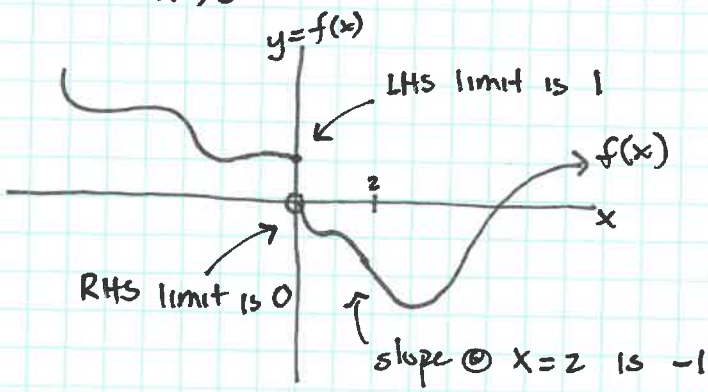
discontinuous @ $x=-2$ and $x=1$

(b) Continuous but NOT differentiable means that the graph has a sharp point. or cusp

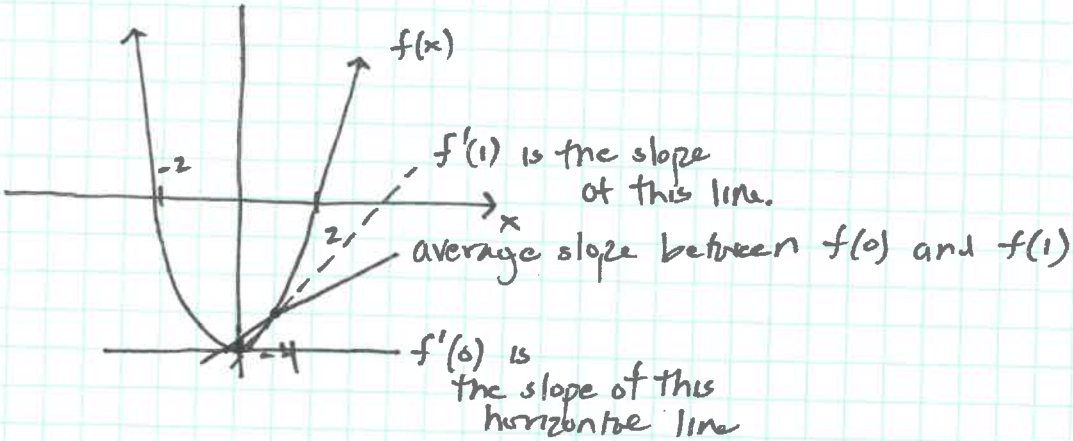


not differentiable at $x=1$ but continuous everywhere.

(c) $\lim_{x \to 0^+} f(x) = 0$ $\qquad \lim_{x \to 0^-} f(x) = 1$ $\qquad f'(2) = -1$



(d)



The derivative is the slope of a tangent line the average rate of change is the slope connecting two points on the graph.

Problem 2 (5 points)

Assume that $f(t) = (t+1)^2$ gives the position of an object at time t. Assume units of meters and seconds.

(a) Find the average velocity between $t = 1$ and $t = 2$.

(b) Find the instantaneous velocity at $t = 1$.

(c) Find the time when the velocity is exactly zero.

(d) For what times is the velocity increasing or decreasing?

(a) $$AV_{[1,2]} = \frac{f(2) - f(1)}{2-1} = \frac{3^2 - 2^2}{1} = \frac{9-4}{1} = 5 \text{ m/s}$$

show the work

include units

(b) $$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h+1)^2 - (1+1)^2}{h}$$

show all work and actually take the limit

$$= \lim_{h \to 0} \frac{((h+2)^2 - 4)}{h} = \lim_{h \to 0} \frac{h^2 + 2h + 4 - 4}{h}$$

$$= \lim_{h \to 0} h + 2 = 2 \text{ m/s}$$

include units

(c) velocity = $f'(t)$

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{(t+h+1)^2 - (t+1)^2}{h}$$

$$= \lim_{h \to 0} \frac{(t^2 + th + t + ht + h^2 + h + t + h + 1) - (t^2 + t + t + 1)}{h}$$

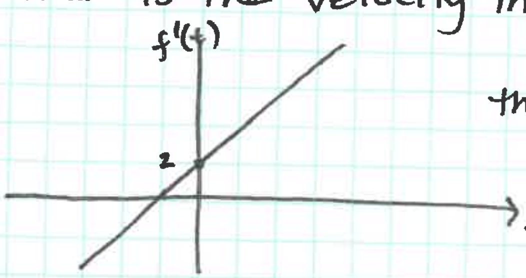
$$= \lim_{h \to 0} \frac{\cancel{t^2} + 2th + \cancel{2t} + 2h + \cancel{1} - \cancel{t^2} - \cancel{2t} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2th + 2h}{h} = \lim_{h \to 0} 2t + 2 = 2t + 2$$

so velocity equals zero when $2t + 2 = 0$

or when $t = -1$ seconds.

(d) When is the velocity increasing? Draw $f'(t)$

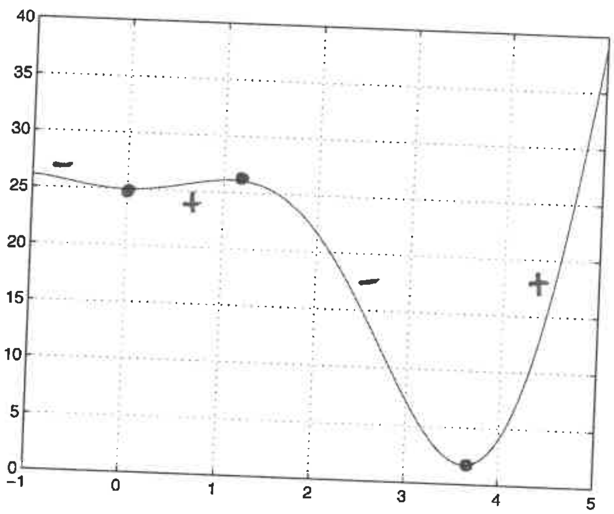


the velocity is always increasing!

slope of velocity is positive.

Problem 3 (5 points)

(a) On the graph below: Put a dot everywhere **the derivative** is zero, a + everywhere the derivative is positive, and a − everywhere the derivative is negative. Explain in words how you decided to make these marks.

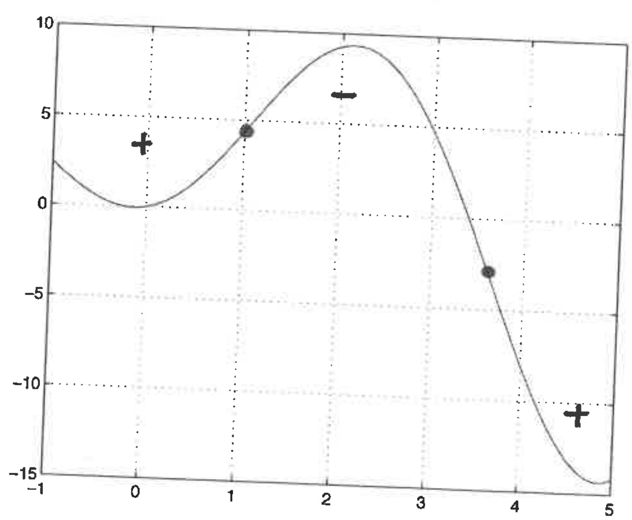


• = $f'(x) = 0$
the slope of the tangent line is zero.

− is where the slope is negative or f(x) is decreasing

+ is where the slope is positive of f(x) is increasing.

(b) On the graph below: Put a dot everywhere **the second derivative** is zero, a + everywhere the derivative is positive, and a − everywhere the derivative is negative. Explain in words how you decided to make these marks.



• = $f''(x) = 0$
This is where the function changes from concave up to concave down (or vice versa)

− is where the function is concave down the slope is decreasing

+ is where the function is concave up the slope is increasing.

Problem 4 (5 points)

The function $G(c)$ represents the average grade on this test as a function of the number of the number of cookies that you bake for your professor (to eat while she's grading). Here c is the number of cookies and G is the average grade measured in percent. For example, $G(1)$ is the average grade on the exam if you baked one cookie for your professor. Explain the meaning of the following:

(a) $G(11) = 90$
(b) $G'(c)$, include units.
(c) $G'(11) = 1$, include units.
(d) Estimate the value of $G(12)$ using the information above.

(a) $G(11) = 90 \Rightarrow$ If we bake joanna 11 cookies the average grade on the exam will be a 90%

(b) $G'(c)$ will have units of %/cookies. It is the change in the average grade over the change in cookies. AKA How much the grade will change if we increase the cookies.

(c) $G'(11) = 1$ The rate of grade change per cookie is 1% per cookie.

(d) $G(12) \sim 91\%$ If we bake one more cookie the average grade goes up by 1%

so $G(11) = 90$ $G'(11) = 1$ means $G(12) \sim 91$

Problem 5 (5 points)
Consider the function $f(x) = x^2 + x$

(a) Find the value of $f'(3)$ **numerically or computationally** by setting up a table of values to evaluate the limit.

(b) Find estimate the value of $f'(3)$ using the **central difference approximation**.

(c) Find the derivative function $f'(x)$ **algebraically**, using the limit definition of the derivative. What is the true value of $f'(3)$? How close were your numerical and estimated values?

(d) Find the second derivative function $f''(x)$, using the limit definition of the second derivative.

$f(x) = x^2 + x.$

(a) Formula: $f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$

TABLE (using a calculator or spreadsheet)

$h = .1$	$f'(3) \sim 7.1$
$h = .01$	$f'(3) \sim 7.01$
$h = .001$	$f'(3) \sim 7.001$
AND	
$h = -.1$	$f'(3) \sim 6.9$
$h = -.01$	$f'(3) \sim 6.99$
$h = -.001$	$f'(3) \sim 6.999$

From both sides we see

$$\boxed{f'(3) = 7}$$

(b) Centre Difference: $f'(3) \sim \frac{f(3+h) - f(3-h)}{2h}$

Choose h small — I pick $h = .001$

then $\boxed{f'(3) \sim 7}$

really I get 6.999999....

(c) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + (x+h)] - [x^2 + x]}{h}$

$= \lim_{h \to 0} \frac{[\cancel{x^2} + 2xh + h^2 \cancel{+x} + h] - \cancel{x^2} \cancel{-x}}{h}$

$= \lim_{h \to 0} \frac{h[2x + 1 + h]}{h} = \lim_{h \to 0} 2x + 1 + h = 2x + 1$

If $\boxed{f'(x) = 2x + 1}$ Then $f'(3) = 2 \cdot 3 + 1 = \boxed{7}$ matches my estimates.

(d) $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[2(x+h) + 1] - [2x + 1]}{h}$

$= \lim_{h \to 0} 2h/h = 2$ so $\boxed{f''(x) = 2}$

Problem 6 (5 points)
Given the function $f(x)$ in the picture below, find the x-value where the following things are true and say in words why you chose that value.

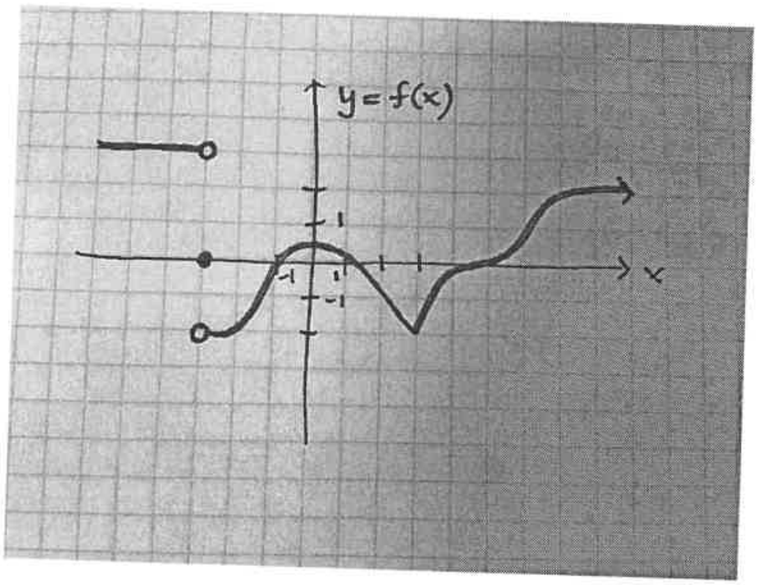
(a) A location where the limit of $f(x)$ does not exist.

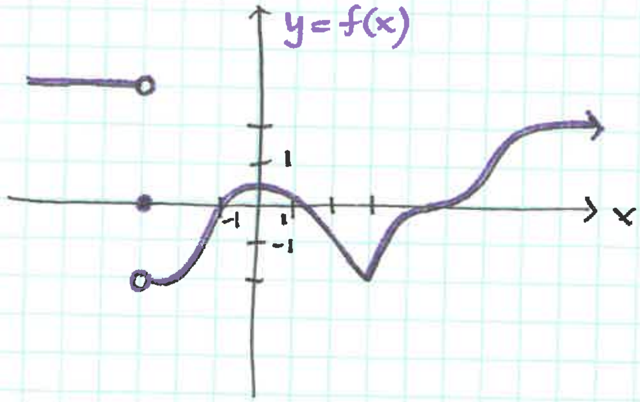
(b) A location where $f'(x) = 0$

(c) A location where the function is continuous but not differentiable

(d) A location where $f''(x)$ is negative

(e) A two points a and b where the average rate of change of $f(x)$ is $AV_{[a,b]} = 2$





(a) There is one location where the limit does not exist

① $x = -3$ $\qquad \lim_{x \to -3^-} f(x) = 3 \qquad \lim_{x \to -3^+} f(x) = -2$

since they don't match up $\lim_{x \to 3} f(x) = DNE.$

(b) The derivative is zero when the slope is zero

so when $x=0$ $\qquad f'(x)=0$

also it looks like $f'(x)=0$ for any $x<-3$
and for large values of x.

(c) At $x=3$ the function is continuous – I can draw it without picking up my pen. But since there is a sharp point, the derivative does not exist

$$f'(3) = DNE.$$

(d) $f''(x) < 0$ means the function is concave down
at $x=0$ the function is concave down.

(e) If I calculate $AV_{[a,b]}$ then I am looking at the slope of the line going throug a and b.

$a=3$ and $b=4$ would work

– OR –

$a=-2$ and $b=-1$ would work.

Problem 7 (5 points)
Every morning I feed my cats Bella and Bongo breakfast. I bring the food over, ask them to sit (which they do!) and then put the two identical bowls down. The following data was taken as I watched my cats eat their morning food. I measured the height of the food in the bowl as a function of time.

Bella's Food

Time t (seconds)	0	10	20	30	40	50	60
Height HBE(t) (centimeters)	4	3	2.25	1.75	1.5	1	.9

Bongo's Food

Time t (seconds)	0	10	20	30	40	50	60
Height HBO(t) (centimeters)	4	2	1.5	1.5	1.5	1.5	1.5

Please answer the following questions:

(a) Are the derivatives $HBE'(t)$ and $HBO'(t)$ ever positive or negative or zero? Explain what each would mean.

(b) What are the units of the derivative? What does the derivative mean in terms of the real world cats.

(c) Estimate each derivative using the method of your choice

$HBE'(10)$

$HBO'(10)$

$HBE'(50)$

$HBO'(50)$

(d) What do the functions and derivatives mean in terms of each cat's eating style? Who eats faster? Who eats more?

(e) Estimate using the derivative at $t = 50$

$HBE(70)$

and

$HBO(70)$

explain why you think your estimate is a good one.

(a) It looks like $HBE'(t)$ is always negative — bella's food is always decreasing

$HBO'(t)$ is negative until $t = 20$ and then zero when he stops eating.

$HBO'(t)$ or $HBE'(t)$ positive would mean that I am refilling the bowl. this is not in the data.

(b) The units of the derivative are cm/s this is

the speed that the cats are eating or how quickly the food in the bowl is decreasing height

(c) I would suggest doing CENTRAL DIFFERENCES.

$$HBE'(10) = \frac{HBE(20) - HBE(0)}{20} = \frac{2.25 - 4}{20} = -.0875 \text{ cm/s}$$

$$HBO'(10) = \frac{HBO(20) - HBO(0)}{20} = \frac{1.5 - 4}{20} = -.125 \text{ cm/s}$$

$$HBE'(50) = \frac{.9 - 1.5}{20} = -.03 \text{ cm/s}$$

$$HBO'(50) = \frac{1.5 - 1.5}{2} = 0 \text{ cm/s}$$

(d) Bongo eats faster but stop eating
Bella east slower but ends up eating more food.

$$HBE(70) \sim HBE(50) + HBE'(50) * 20 = 1 + 20(-.03) = .4 \text{ cm}$$

$$HBO(70) \sim HBO(50) + HBO'(50) \times 20 = 1.5 + 20(0) = 1.5$$

This works because I use the food height at $t = 50$ and then assume the rate will remain the same over the next 20 seconds.

mult by 20 because HBO' is cm/s and there are 20 seconds between 50 and 70.