Day 4 Calc II

continue integration by parts.

Today we will do more examples using integration by parts and learn some tricks for integrating sine and cosine.

Ex $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

Function inside function $\Rightarrow$ integration by substitution.

let $w = \sqrt{x}$ then $dw = \dfrac{dx}{2\sqrt{x}}$

$dx = 2\sqrt{x}\, dw$

$$\int \sqrt{x}\, e^{w}\, 2\sqrt{x}\, dw = \int 2x e^{w} dw = \int 2w^2 e^{w} dw$$

$x = w^2$ from the substitution.

No we have $2\int w^2 e^{w} dw$

Two functions multiplied $\Rightarrow$ integration by parts.

How many times? TWICE.

YOU TRY
$\int x^8 \sin(x^3)\, dx$
$\int x e^{5x}\, dx$

The Table method...

let $u = w^2$ and $dv = e^{w}$

take derivatives $\downarrow$ | integrate $\downarrow$

w^2	$+$	e^{w}
$2w$	$-$	e^{w}
2	$+$	e^{w}
0		e^{w}

$$2\int w^2 e^{w} dw = 2\left[w^2 e^{w} - 2we^{w} + 2e^{w}\right] + c$$

Final answer $\int \sqrt{x}\, e^{\sqrt{x}}\, dx = 2\left[x e^{\sqrt{x}} - 2\sqrt{x}\, e^{\sqrt{x}} + 2e^{\sqrt{x}}\right] + c$

- Making a copy of the original integral – using integration by parts once or twice.

Ex $\int \cos^2\theta \, d\theta$

let $u = \cos\theta$ $dv = \cos\theta \, d\theta$
then $du = -\sin\theta \, d\theta$ $v = \sin\theta$

$$= \cos\theta \sin\theta + \int \sin^2\theta \, d\theta$$

but $\sin^2\theta + \cos^2\theta = 1$ Trig Identity
or $\sin^2\theta = 1 - \cos^2\theta$.

$$= \cos\theta \sin\theta + \int 1 - \cos^2\theta \, d\theta$$

so

$$\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \int d\theta - \int \cos^2\theta \, d\theta$$

Solve for the original integral.

$$2\int \cos^2\theta \, d\theta = \cos\theta \sin\theta + \theta + C$$

$$\boxed{\int \cos^2\theta \, d\theta = \frac{1}{2}\cos\theta \sin\theta + \frac{\theta}{2} + C}$$

Important Trig Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}\left[1 - \cos 2\theta\right]$$
$$\cos^2\theta = \frac{1}{2}\left[1 + \cos 2\theta\right]$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Memorize These !!

Ex $\int e^x \cos 2x \, dx$ Integration by parts twice
make a copy of original integral.

let $u = e^x$ $dv = \cos 2x \, dx$
$du = e^x dx$ $v = \frac{1}{2}\sin 2x$

$$= \frac{e^x}{2}\sin 2x - \frac{1}{2}\int e^x \sin 2x \, dx$$

let $u = e^x$ and $dv = \sin 2x \, dx$
$du = e^x dx$ $v = -\frac{1}{2}\cos 2x$

$$= \frac{e^x}{2}\sin 2x - \frac{1}{2}\left[-\frac{e^x}{2}\cos 2x + \frac{1}{2}\int e^x \cos 2x \, dx\right]$$

$$\int e^x \cos 2x = \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x - \underbrace{\frac{1}{4}\int e^x \cos 2x \, dx}_{\text{this matches the original integral.}}$$

$$\frac{5}{4}\int e^x \cos 2x = \frac{e^x}{2}\sin 2x + \frac{e^x}{4}\cos 2x$$

$$\boxed{\int e^x \cos 2x = \frac{2}{5}e^x \sin 2x + \frac{e^x}{5}\cos 2x + C}$$

You try: $\int e^{2x} \sin x \, dx.$

A GENERAL RULE FOR INTEGRATING POWERS of SINE + COSINE.

Even Powers.
use the identity
$\sin^2\theta = \frac{1}{2}[1 - \cos 2\theta]$
$\cos^2\theta = \frac{1}{2}[1 + \cos 2\theta]$
$\longrightarrow$

$$\int \sin^2\theta \, d\theta = \int \frac{1}{2}[1 - \cos 2\theta] \, d\theta$$
$$= \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta$$
$$= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

Ex $\int \cos^4\theta \, d\theta = \int \cos^2\theta \cdot \cos^2\theta \, d\theta = \int \frac{1}{2}[1+\cos 2\theta] \cdot \frac{1}{2}[1+\cos 2\theta] \, d\theta$

break into squares

$= \frac{1}{4}\int (1+\cos 2\theta)(1+\cos 2\theta)\, d\theta = \frac{1}{4}\int 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$

foil

$= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{4}\int \cos^2 2\theta \, d\theta$

do it again

$= \frac{\theta}{4} + \frac{1}{4}\sin 2\theta + \frac{1}{4}\int \frac{1}{2}(1+\cos 4\theta)\, d\theta$

$= \frac{\theta}{4} + \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{8}\cdot\frac{1}{4}\sin 4\theta + c$

$= \frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c.$

Odd Powers

Use the identity

$\sin^2\theta + \cos^2\theta = 1$

then substitution $\longrightarrow$

$\int \sin^3\theta \, d\theta = \int \sin\theta \sin^2\theta \, d\theta$

$= \int \sin\theta (1-\cos^2\theta)\, d\theta$

let $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$= \int -(1-u^2)\, du = \frac{1}{3}u^3 - u + c$

$= \frac{1}{3}\cos^3\theta - \cos\theta + c$