**Problem 1** (0 points)

Find the Taylor Series centered at $c = 0$ for the function $f(x) = \cos(x^2)$.

Now integrate this series to find $\int_0^1 \cos(x^2)\,dx$. Your answer should be an infinite series.

Now approximate this integral using the first four terms in the series you just found.

**More Practice:** Redo the problem for $f(x) = \frac{\cos(x)}{x}$ and $f(x) = \sin(x^3)$.

Problem 1

Expand $f(x) = \cos(x^2)$

use $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$

let $u = x^2$ then

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2n!}$$

Then integrate term by term:

$$I = \int_0^1 \left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right] dx = x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \cdots \Big|_0^1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}$$

$$I \approx x - \frac{x^5}{5 \cdot 2} + \frac{x^9}{9 \cdot 24} - \frac{x^{13}}{13 \cdot 720} \Big|_0^1 = 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} = 0.9045228$$

MORE PRACTICE...

$$f(x) = \frac{\cos(x)}{x} = \frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots$$

$$I = \int_0^1 \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots dx$$

$$= \underbrace{\int_0^1 \frac{1}{x} dx}_{\text{improper}} + \underbrace{\int_0^1 \frac{-x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots dx}_{\text{evaluate term by term.}}$$

$$\int_a^1 \frac{1}{x} dx = \ln|x| \Big|_a^1 = \ln|1| - \ln|a| = -\ln|a|$$

$$\lim_{a \to 0} -\ln|a| = +\infty \quad \text{Diverges!}$$

so $\int_0^1 \frac{\cos(x)}{x} dx$ diverges. can't approximate.

MORE PRACTICE...

Expand $f(x) = \sin(x^3)$

use $\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$

then $u = x^3$

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

Then integrate term by term.

$$I = \int_0^1 x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots \, dx = \frac{x^4}{4} - \frac{x^{10}}{10 \cdot 3!} + \frac{x^{16}}{16 \cdot 5!} - \frac{x^{22}}{22 \cdot 7!} + \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(6n+4)(2n+1)!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+4)(2n+1)!}$$

$$I \approx \frac{1}{4} - \frac{1}{60} + \frac{1}{1920} - \frac{1}{110880} = 0.2338451479$$

**Problem 2** (0 points)

Consider the sequence $a_n = \frac{n}{n^2-1}$

(a) Does the **sequence** $a_n$ converge or diverge? What does the limit as $n \to \infty$ of this sequence tell you about $\sum_{n=0}^{\infty} a_n$? What does it tell you about $\sum_{n=0}^{\infty}(-1)^n a_n$?

(b) Make a conjecture: Does the series $\sum_{n=0}^{\infty} a_n$ converge or diverge?

(c) Prove, using a comparison or limit comparison test, your conjecture from part (b).

(d) What does the ratio test tell you about $\sum_{n=0}^{\infty} a_n$?

**More Practice:** Redo the problem for $a_n = \frac{1}{3^n+1}$, $a_n = \frac{1}{n^4+3n^3+7}$, and $a_n = \frac{5n+1}{3n^2-1}$

Problem 2 $a_n = \frac{n}{n^2+1}$

(a) the sequence goes $0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} \cdot \left(\frac{1}{1+n^{-2}}\right) = 0$$

so the sequence converges to zero

This tells me that

$\sum_{n=1}^{\infty} a_n$ might converge and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges at least conditionally

(b) $\sum_{n=1}^{\infty} a_n$ is going to act like $\frac{1}{n}$ which diverges.

guess diverge.

(c) Use limit comparison test

$a_n = \frac{n}{n^2+1}$ and $b_n = \frac{1}{n}$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

so $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

this is a p-series $p=1$, diverges

so

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

(d) Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2}\right| = 1$

ratio test tells us nothing.

$a_n = \frac{1}{3^n+1}$

(a) $\lim_{n\to\infty} a_n = 0$ $a_n$ converges to zero

so $\sum_{n=1}^{\infty} a_n$ might converge

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges

(b) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ behaves like $\left(\frac{1}{3}\right)^n$ as $n$ gets large

this is like the geometric series which converges when $|x|<1$

(c) Compare $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ to $\sum_{n=1}^{\infty} \frac{1}{3^n}$

since $\sum_{n=1}^{\infty} \frac{1}{3^n+1} < \sum_{n=1}^{\infty} \frac{1}{3^n}$ and $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges

it is the geometric series $x = 1/3 < 1$

so $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ also converges.

(d) Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n+1}{3^{n+1}+1}\right| = \lim_{n\to\infty} \left|\frac{3^n(1+3^{-n})}{3^n(3+3^{-n})}\right|$$

$$= \lim_{n\to\infty} \left|\frac{1+3^{-n}}{3+3^{-n}}\right| = \frac{1}{3} < 1$$

thus $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ converges absolutely.

$a_n = \dfrac{1}{n^4+3n^3+7}$ (a) $\lim\limits_{n\to\infty} \dfrac{1}{n^4+3n^3+7} = 0$

so the sequence $a_n$ converges to zero.

$\sum\limits_{n=1}^{\infty} a_n$ might converge $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ converges.

(b) I think $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4+3n^3+7}$ will converge.

because as $n$ get large it behaves as $1/n^4$.

(c) Use the limit comparison test

$a_n = \dfrac{1}{n^4+3n^3+7}$ $b_n = \dfrac{1}{n^4}$

then $\lim\limits_{n\to\infty} \left| \dfrac{a_n}{b_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{n^4}{n^4+3n^3+7} \right|$

$= \lim\limits_{n\to\infty} \left| \dfrac{n^4}{n^4} \cdot \dfrac{1}{1+3n^{-1}+7n^{-4}} \right| = \lim\limits_{n\to\infty} \left| \dfrac{1}{1+3n^{-1}+7n^{-4}} \right| = 1$

so since

$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$ converges so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4+3n^3+7}$.

(d) Ratio Test

$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{n^4+3n^3+7}{(n+1)^4+3(n+1)^3+7} \right| = 1$ tells me nothing.

**Problem 3** (0 points)

For the power series given below, find the Radius of Convergence and the Interval of Convergence.

$$\sum_{n=0}^{\infty} nx^n$$

**More Practice:** $\sum_{n=0}^{\infty}(2^n + n^2)x^n$ and $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n2^n}$

Problem 3

$$\sum_{n=0}^{\infty} n x^n$$

radius of convergence

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}x\right| = |x| < 1$$

Radius of convergence $R=1$

Interval of convergence $-1 < x < 1$

MORE PRACTICE

$$\sum_{n=0}^{\infty}(2^n+n^2)x^n$$

Ratio Test $a_{n+1} = (2^{n+1}+(n+1)^2)x^{n+1} = (2^{n+1}+n^2+2n+1)x^{n+1}$

$a_n = (2^n+n^2)x^n$

$$\lim_{n\to\infty}\left|\frac{2^{n+1}+n^2+2n+1}{2^n+n^2}\;\frac{x^{n+1}}{x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{2^{n+1}+n^2+2n+1}{2^n+n^2}\right|$$

l'Hopitals: take derivatives $\dfrac{2\ln(2)2^n+2n+2}{\ln(2)2^n+2n}$

do it again $\dfrac{2\ln(2)^2 2^n+2}{\ln(2)^2 2^n+2} = 2$

and $\lim_{n\to\infty} 2 = 2$

Ratio Test gives

$2|x|<1$ so $|x|<\frac{1}{2}$ $R=1/2$ Radius of Convergence

and $-1/2 < x < 1/2$ is the Interval of Convergence.

MORE PRACTICE

$$\sum_{n=0}^{\infty} \frac{1}{n2^n}(x-1)^n \qquad \text{centered } c=1$$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n\,2^n}{(n+1)2^{n+1}}\frac{(x-1)^{n+1}}{(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{1}{2}(x-1)\right| = \frac{1}{2}|x-1| < 1$$

$$|x-1| < 2 \qquad -2 < x-1 < 2$$

$$-1 < x < 3$$

Radius of Convergence $R=2$

Interval of Convergence

$-1 < x < 3$

**Problem 4** (0 points)

Consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$$

(a) What does the Leibniz Test say about the infinite series? Explain your answer.

(b) Does the series converge absolutely? Show mathematical work to back up your answer.

Problem 4

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{1/3}}$$

Leibniz Test
(Alternating Series Test)

Since $a_n = \frac{1}{n^{1/2}} \to 0$
as $n \to \infty$ this
series converges...

Check for absolute convergence

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^{1/3}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$ this diverges it is the p-series
with $p = 1/3$, diverges.

So the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$ converges conditionally.

**Problem 5** (0 points)
Use the integral test to prove whether the following series converge or diverge.

A. $\sum_{n=0}^{\infty} \frac{1}{(n+2)^2}$

For the integral test we compare to the improper integral.

B. $\sum_{n=0}^{\infty} \frac{1}{e^n}$

A. $\int_0^{\infty} \frac{1}{(x+2)^2}\,dx$ consider $\int_0^{b} \frac{1}{(x+2)^2}\,dx = \int_2^{b+2} \frac{1}{u^2}\,du$

let $u = x+2$
$du = dx$
$x=0 \quad u=2$
$x=b \quad u=b+2$

$$= \frac{-1}{u}\Big|_2^{b+2} = \frac{-1}{b+2} + \frac{1}{2} = \frac{1}{2} - \frac{1}{b+2}$$

$$\lim_{b\to\infty} \frac{1}{2} - \frac{1}{b+2} = 1/2$$

the Integral Converges so the Series $\sum_{n=0}^{\infty} \frac{1}{(n+2)^2}$ converges.

B. $\int_0^{\infty} e^{-x}\,dx$ consider $\int_0^{b} e^{-x}\,dx = -e^{-x}\Big|_0^b = -e^{-b} + e^0$
$= 1 - e^{-b}$

$$\lim_{b\to\infty} 1 - e^{-b} = 1$$

the integral Converges so the Series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges.

**Problem 6** (0 points)

(a) Show that the Taylor Series for $f(x) = e^x$ is given by $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

(b) Write down the Taylor Polynomial centered at $x = 0$ that approximates $e^{-0.2}$ to six decimal places. You will need to use the Lagrange Error Bound and a calculator.

(c) Use the Taylor Series in part (a) to solve the following integral:

$$\int_0^1 e^{-x^2}\,dx$$

Use the formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

centered at $x=0 \rightarrow a=0$

Take Derivatives

| | |
|---|---|
| $f(x) = e^x$ | $f(0) = 1$ |
| $f'(x) = e^x$ | $f'(0) = 1$ |
| $f''(x) = e^x$ | $f''(0) = 1$ |
| $\vdots$ | $\vdots$ |

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \ldots$$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Error Bound

$$|E_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

$f^{(n)}(x) = e^x$

near $x = -0.2$

$e^{-0.2} < e^0 < 1$

choose $M = 1$

$$= \frac{1}{(n+1)!}|x|^{n+1} = \frac{|.2|^{n+1}}{(n+1)!} < 10^{-6}$$

find the n-value: $n = 5$ works.

to solve the integrul:

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

then

$$\int_0^1 e^{-x^2} dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, n!}$$