MATH 122
Practice Exam B

Joanna Bieri

Name: Solutions.

Instructions:
This is a set of practice problems for Exam 1.

If you are able to do all these problems without looking at books, notes, OR SOLUTIONS then you should be prepared for the exam. You should know that these problems are representative of the type of problems on the exam, not just a copy of the exam problems with some constants changed. You must understand the underlying methods to do well on the exam.

Also make sure you can do all the homework problems and example problems from the class notes.

Important Ideas:

1. General Integration Methods from Chapter 7 - Substitution, Integration by Parts, Trigonometric Substitution, Algebraic Simplification, Powers of sin(x) and cos(x), Methods of Approximation.
2. Finding Area and Volume using Integration.
3. Volumes of Revolution.
4. Applications to Physics: Mass, Work, and Pressure.
5. Applications to Economics.

Score

1	/30
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total	/100

Problem 1 (30 points)

Complete the square and solve the resulting integral.

$$\int \frac{1}{y^2+2y+5}\,dy$$

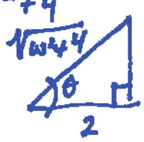
$y^2+2y+5 = (y+b)^2+c$
$= y^2+2by+b^2+c$

$2b=2 \quad b=1$
$b^2+c=5$
$c=5-b^2=5-1=4$

[HINT: You will need to use a trigonometric substitution to evaluate the integral, make sure to draw your triangle]

$= \int \frac{1}{(y+1)^2+4}\,dy = \int \frac{1}{w^2+4}\,dw \longrightarrow$

let $w=y+1$
$dw=dy$



$\tan\theta = \frac{w}{2}$
$w = 2\tan\theta$
$dw = \frac{2d\theta}{\cos^2\theta}$
$\cos\theta = \frac{2}{\sqrt{w^2+4}}$

$= \int \frac{\cos^2\theta}{4}\cdot\frac{2d\theta}{\cos^2\theta} = \frac{1}{4}\int d\theta$

$= \frac{1}{4}\theta + c = \frac{1}{4}\arctan\left(\frac{w}{2}\right)+c$

$= \frac{1}{4}\arctan\left(\frac{y+1}{2}\right)+c$

Evaluate the following integrals using integration by parts.

$$\int x\ln(x)\,dx \longrightarrow \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx =$$

let $u=\ln(x)$ $dv = x\,dx$
$du = \frac{1}{x}dx$ $v=\frac{1}{2}x^2$

$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + c$

Evaluate the following integral using substitution.

$$\int_0^1 \frac{\cos(x)}{1-\sin(x)}\,dx \longrightarrow \int_0^{\sin(1)} \frac{1}{1-u}\,du = -\ln|1-u|\Big|_0^{\sin(1)}$$

let $u=\sin(x)$
$du = \cos(x)dx$
$dx = \frac{du}{\cos(x)}$

when $x=0$ $u=\sin(0)=0$
$x=1$ $u=\sin(1)$

$= -\ln|1-\sin(1)| + \cancel{\ln|1|}^{0}$

$= -\ln|1-\sin(1)|$

Problem 2 (10 points)

Evaluate the following improper integral.

$$\int_1^3 \frac{1}{(x-2)^2}\,dx = \int_1^2 \frac{1}{(x-2)^2}\,dx + \int_2^3 \frac{1}{(x-2)^2}\,dx$$

consider:

$$\int_1^b \frac{1}{(x-2)^2}\,dx = \left.\frac{-1}{x-2}\right|_1^b = \frac{-1}{b-2} + \frac{1}{1-2} = \frac{-1}{b-2} - 1$$

$$\int_b^3 \frac{1}{(x-2)^2}\,dx = \left.\frac{-1}{x-2}\right|_b^3 = \frac{-1}{3-2} + \frac{1}{b-2} = -1 + \frac{1}{b-2}$$

$$\int_1^2 \frac{1}{(x-2)^2}\,dx = \lim_{b \to 2^-} \frac{-1}{b-2} - 1 = \infty - 1 \quad \text{undefined}$$

$$\int_2^3 \frac{1}{(x-2)^2}\,dx = \lim_{b \to 2^+} -1 - \frac{1}{b-2} = -1 - \infty \quad \text{undefined.}$$

The integral diverges.

Problem 3 (10 points)

Evaluate the following indefinite integral using the method of your choice. You must state what method you are using!

$$\int \frac{cos(x)}{1+sin(x)}\,dx \xrightarrow{\text{Substitution.}} \int \frac{1}{1+w}\,dw = \ln|1+w| + c$$

$$= \ln|1+\sin x| + c$$

let $w = \sin x$

$dw = \cos x\,dx$

$dx = \frac{dw}{\cos x}$

Evaluate the following definite integral using the method of your choice. You must state what method you are using!

$$\int_0^1 te^{2t}\,dt \xrightarrow{\text{integration by parts.}} \frac{t}{2}e^{2t}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2t}\,dt$$

let $u = t$ $\quad dv = e^{2t}dt$

$du = dt$ $\quad v = \frac{1}{2}e^{2t}$

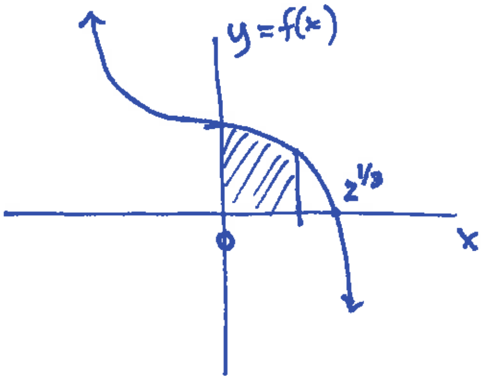
$$= \frac{1}{2}e^2 - \frac{1}{2}\cdot\frac{1}{2}e^{2t}\Big|_0^1$$

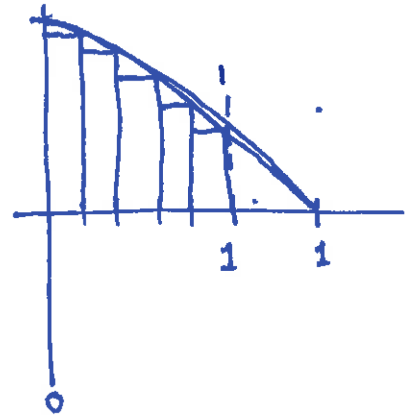
$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}\left[1+e^2\right]$$

Problem 4 (10 points)

Sketch the function $f(x) = -x^3 + 2$, then answer the following questions:





a. How would you use ~~the midpoint rule~~ RIGHT(5) to evaluate $\int_0^1 f(x)\,dx$? Sketch ~~MID~~ Right(5) on your graph of $f(x)$.

Divide the region into 5 boxes taking the height of the box from the right hand side.

b. Would ~~MID~~ Right(5) be an underestimate or overestimate of the value of the integral?

It would be an underestimate because the boxes are all beneath the curve.

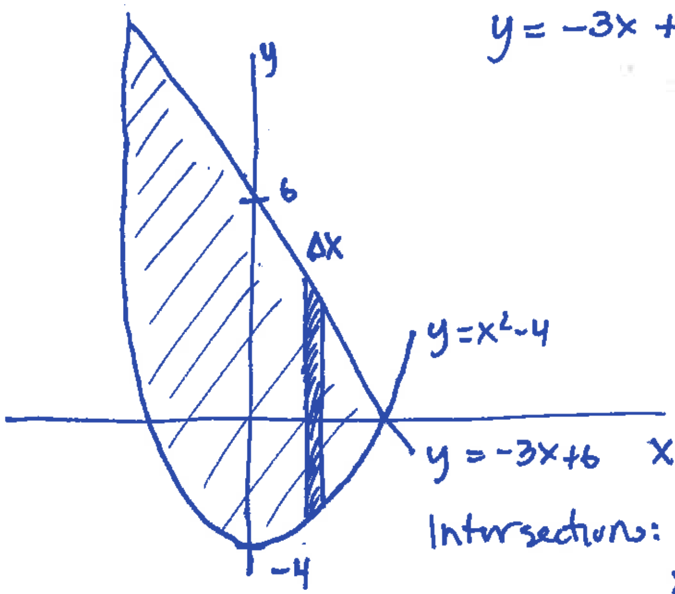
c. How would you find ~~SIMP~~ MID(5)? (Just say in words what you would have to do.)

Instead take the height of the box from a point in the middle of the range.

Problem 5 (10 points)

Draw the area bounded by the curves $3x + y = 6$ and $y = x^2 - 4$, then use integration to find the value for the area bounded by the two curves.

$y = -3x + 6$



Slice:

$A_i = -3x+6-x^2+4 \; \Delta x$

add all the pieces.

Intersections: $x^2 - 4 = -3x+6$

$x^2+3x-10 = 0 \quad (x+5)(x-2)=0$

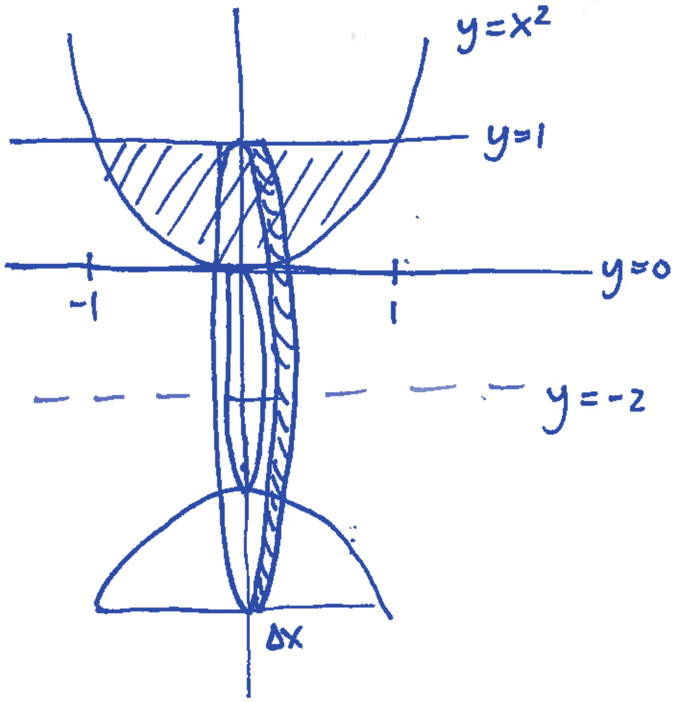
$$\text{Area}_T = \lim_{n\to\infty} \sum_{i=1}^{n} -3x_i+10-x_i^2 \,\Delta x = \int_{-5}^{2} 10 - 3x - x^2 \, dx$$

$$= 10x - \frac{3}{2}x^2 - \frac{x^3}{3}\Big|_{-5}^{2} = 10(2) - 10(-5) - \frac{3}{2}(2)^2 + \frac{3}{2}(-5)^2 - \frac{(2)^3}{3} - \frac{(-5)^3}{3}$$

$$= 20 + 50 + -6 + \frac{3(25)}{2} - \frac{8}{3} + \frac{125}{3}$$

<u>Problem 6</u> (10 points)

Find the volume of the solid generated by rotating the region bounded by $y = x^2$, $y = 0$, $y = 1$ about the line $y = -2$.



Slice:

$$V_i = \pi(r_2^2 - r_1^2)\Delta x$$

$$r_2 = 1 - (-2) = 3$$

$$r_1 = x^2 - (-2) = x^2 + 2$$

$$V_i = \pi\left((x^2+2)^2 - 9\right)\Delta x$$

$$V_{TOTAL} = \pi \int_{-1}^{1} \left[(x^2+2)^2 + 9\right] dx = \pi \int_{-1}^{1} -x^4 - 4x^2 - 4 + 9\, dx$$

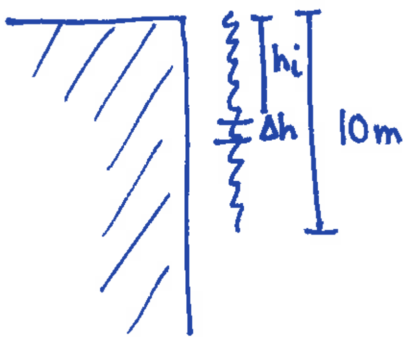
$$= -\pi \int_{-1}^{1} x^4 + 4x^2 - 5\, dx = -\pi\left[\frac{x^5}{5} + \frac{4x^3}{3} - 5x\right]\Bigg|_{-1}^{1}$$

$$= -\pi\left[\frac{1}{5} + \frac{4}{3} - 5\right] + \pi\left[\frac{-1}{5} - \frac{4}{3} + 5\right]$$

$$= \pi\left[\frac{-2}{5} - \frac{8}{3} + 10\right]$$

Problem 7 (10 points)

A 10 meter uniform chain with a mass of 5 kilograms per meter is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building? (acceleration of gravity: $9.8\frac{m}{s^2}$



mass = 5 kg/m

work

$I \ \Delta h \quad W_i = \text{Force} \cdot \text{Distance}$

$= 5(9.8)\Delta h \quad h_i$

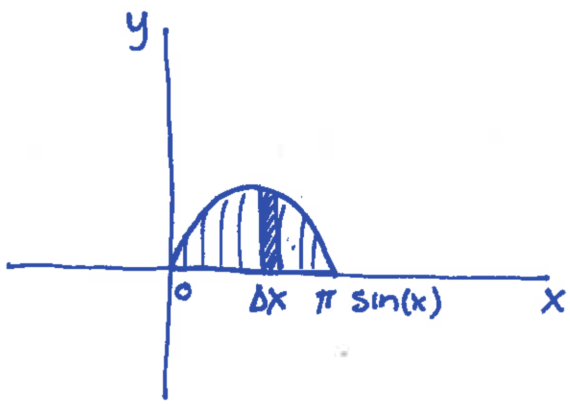
$= 5(9.8) h_i \Delta h.$

$$W_{TOTAL} = 5(9.8)\int_0^{10} h\,dh = 5(9.8)\left.\frac{h^2}{2}\right|_0^{10}$$

$$= 5(9.8)\frac{10^2}{2} = \frac{5}{2} \cdot 98 \text{ J}$$

Problem 8 (10 points)

Find the mass of the region bounded by $y = \sin(x)$ and $y = 0$ between $x = 0$ and $x = \pi$, if the density is $\delta(x) = x\frac{g}{cm^3}$.



M_i = Density · Area
$= x_i \sin(x) \Delta x$

$$\text{Mass}_{\text{TOTAL}} = \int_0^{\pi} x\sin(x)\,dx$$

let $u = x$ $\quad dv = \sin(x)\,dx$
$du = dx$ $\quad v = -\cos(x)$

$$= -x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx$$

$$= -\pi\cos(\pi) + \sin(x)\Big|_0^{\pi}$$

$$= \pi + \sin(\pi) - \sin(0) = \pi \text{ grams.}$$