From Last time ...

More Improper Integrals.

Ex. $\int_0^3 \frac{1}{(x-2)^2}dx$ problem at $x=2$

$$= \int_0^2 \frac{1}{(x-2)^2}dx + \int_2^3 \frac{1}{(x-2)^2}dx$$

$$\int_0^b \frac{1}{(x-2)^2}dx = \frac{-1}{x-2}\Big|_0^b = \frac{-1}{b-2} - \frac{-1}{-2} = \frac{-1}{b-2} - \frac{1}{2}$$

$$\int_a^3 \frac{1}{(x-2)^2}dx = \frac{-1}{x-2}\Big|_a^3 = \frac{-1}{1} + \frac{1}{a-2}$$

$$\lim_{b\to 2^-} \frac{-1}{b-2} - \frac{1}{2} = \infty - \frac{1}{2}$$ Diverges

→ why positive ∞ ?
Because we approach from the LEFT

$\frac{-1}{1-2} = +\frac{1}{2}$ $\quad \frac{-1}{1.5-2} = +2 \quad \frac{-1}{1.9-2} = +10$ etc ...

$$\lim_{b\to 2^+} -1 + \frac{1}{a-2} = -1 + \infty$$ Diverges

→ We are approaching from the RIGHT.

The integral $\int_0^3 \frac{1}{(x-2)^2}dx$ diverges.

If you did not recognize this as improper then you would get!..

$$\int_0^3 \frac{1}{(x-2)^2}dx = \frac{-1}{x-2}\Big|_0^3 = \frac{-1}{3-2} - \frac{-1}{0-2}$$

$$= -1 - \tfrac{1}{2} = -\tfrac{3}{2}$$

you would be wrong!

[redacted]

More Improper Integrals

Examples – integration. Tricks and Tips.

Ex $\displaystyle\int_0^6 \frac{1}{(x-4)^{2/3}}\,dx$

trouble spot at $x=4$...

$$\int_0^4 \frac{1}{(x-4)^{2/3}}\,dx + \int_4^6 \frac{1}{(x-4)^{2/3}}\,dx$$

$$= \lim_{b\to 4^-} \int_0^b \frac{1}{(x-4)^{2/3}}\,dx + \lim_{b\to 4^+} \int_b^6 \frac{1}{(x-4)^{2/3}}\,dx.$$

$$\int_0^b \frac{1}{(x-4)^{2/3}}\,dx = 3(x-4)^{1/3}\Big|_0^b = 3(b-4)^{1/3} - 3(-4)^{1/3} = 3(b-4)^{1/3} + 3(4)^{1/3}$$

$$\int_b^6 \frac{1}{(x-4)^{2/3}}\,dx = 3(x-4)^{1/3}\Big|_b^6 = 3(2)^{1/3} - 3(b-4)^{1/3}$$

take the limit...

$$\lim_{b\to 4^-}\left[3(b-4)^{1/3} + 3(4)^{1/3}\right] + \lim_{b\to 4^+}\left[3(2)^{1/3} - 3(b-4)^{1/3}\right]$$

$$= 3(4)^{1/3} + 3(2)^{1/3} = 3\left[(4)^{1/3} + (2)^{1/3}\right]$$

Ex $\displaystyle\int_0^\infty \frac{1}{x^3}\,dx = \int_0^1 \frac{1}{x^3}\,dx + \int_1^\infty \frac{1}{x^3}\,dx$

consider

$$\int_a^1 \frac{1}{x^3}\,dx = -\frac{1}{2}x^{-2}\Big|_a^1 = -\frac{1}{2}\left[1^{-2} - a^{-2}\right] = \frac{1}{2a^2} - \frac{1}{2}$$

$$\int_1^b \frac{1}{x^3}\,dx = -\frac{1}{2}x^{-2}\Big|_1^b = -\frac{1}{2}\left[b^{-2} - 1^{-2}\right] = \frac{1}{2} - \frac{1}{2b^2}$$

$\displaystyle\lim_{a\to 0} \frac{1}{2a^2} - \frac{1}{2} =$ diverges $\qquad \displaystyle\lim_{b\to\infty} \frac{1}{2} - \frac{1}{2b^2} = \frac{1}{2}$

But $\displaystyle\int_0^\infty \frac{1}{x^3}\,dx$ diverges

Special Case...

$$\int_0^3 \frac{1}{(x-1)^3}\,dx = \int_0^1 \frac{1}{(x-1)^3}\,dx + \int_1^3 \frac{1}{(x-1)^3}\,dx$$

consider:

$$\int_0^b \frac{1}{(x-1)^3}\,dx = -\frac{1}{2}(x-1)^{-2}\Big|_0^b = -\frac{1}{2}(b-1)^{-2} + \frac{1}{2}(-1)^{-2}$$

$$\int_a^3 \frac{1}{(x-1)^3}\,dx = -\frac{1}{2}(x-1)^{-2}\Big|_a^3 = -\frac{1}{2}(2)^{-2} + \frac{1}{2}(a-1)^{-2}$$

$\lim_{b\to 1^-}\left[\frac{-1}{2(b-1)^2} + \frac{1}{2}\right]$ diverges $-\infty$

$\lim_{a\to 1^+}\left[\frac{-1}{8} + \frac{1}{2}(a-1)^{-2}\right]$ diverges $+\infty$

BUT the first part goes to $-\infty$ at the same rate that the second goes to $-\infty$

We would say the integral <u>does not converge</u> ... diverges.

Cauchy Value $= \frac{3}{8}$
Principal value

some people use this
we will not in this class.

Review of Trig Sub...

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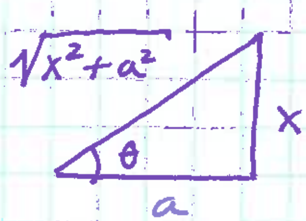
Three Cases

Terms like $(x^2+a^2)^n$ $\longrightarrow$ TANGENT

memorize: $x = a\tan\theta$

plug in and use identities

$\tan^2\theta + \sec^2\theta = 1$



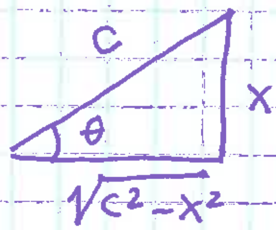
$$\int \frac{dx}{x\sqrt{x^2+16}}$$

Terms like $(a^2-x^2)^n$ $\longrightarrow$ SINE

memorize: $x = c\sin\theta$

plug in and use identities

$\sin^2\theta + \cos^2\theta = 1$



$$\int x^3\sqrt{9-x^2}\,dx$$

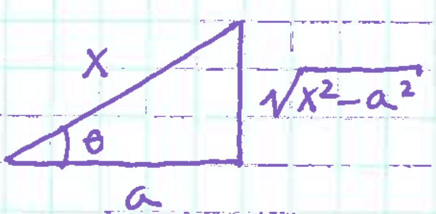
SPECIAL CASE!

Terms like $(x^2-a^2)^n$ $\longrightarrow$ SECANT

memorize: $x = a\sec\theta$

plug in and use identities.

$\tan^2\theta + \sec^2\theta = 1$

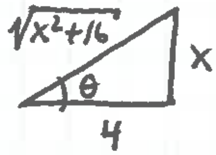


$$\int \frac{dx}{\sqrt{x^2-9}}$$

More examples — harder integrals.

<u>EX</u> $\displaystyle\int \frac{dx}{x\sqrt{x^2+16}}$ looks like a trig sub.

x^2+16 looks like a^2+b^2



$\tan\theta = x/4$

$x = 4\tan\theta$

$dx = \dfrac{4\,d\theta}{\cos^2\theta}$

$\cos\theta = \dfrac{4}{\sqrt{x^2+16}}$

$\displaystyle\int \frac{1}{x}\cdot\frac{1}{\sqrt{x^2+16}}\,dx$ SUB IN

$\dfrac{1}{x} \to \dfrac{1}{4\tan\theta}$, $\dfrac{1}{\sqrt{x^2+16}} \to \dfrac{1}{4}\cos\theta$, $dx \to \dfrac{4\,d\theta}{\cos^2\theta}$

$$\frac{1}{4}\int \frac{1}{\tan\theta}\cos\theta\,\frac{d\theta}{\cos^2\theta} = \frac{1}{4}\int \frac{\cos\theta}{\sin\theta}\cdot\frac{d\theta}{\cos\theta} = \frac{1}{4}\int\frac{1}{\sin\theta}\,d\theta =$$

A trick for integrating $1/\sin\theta$

$$= \frac{1}{4}\int\frac{\sin\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int\frac{\sin\theta}{1\mp\cos^2\theta}\,d\theta = \frac{1}{4}\int\frac{-1}{1-u^2}\,du$$

let $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$d\theta = \dfrac{-du}{\sin\theta}$

$$= \frac{-1}{4}\int\frac{1}{(1+u)(1-u)}\,du$$

integration by parts.

$$\frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

$1 = A(1-u) + B(1+u)$

$= (B-A)u + (A+B)$

$A+B=1 \quad B-A=0 \quad B=A$

$A = 1/2 \quad B = 1/2$

$$= \frac{-1}{4}\int\frac{1/2}{1+u}\,du - \frac{1}{4}\int\frac{1/2}{1-u}\,du = -\frac{1}{4}\Big[\ln|1+u| - \ln|1-u|\Big] + C$$

$$= -\frac{1}{4}\Big[\ln|1+\cos\theta| - \ln|1-\cos\theta|\Big] + C = -\frac{1}{4}\left[\ln\left|1+\frac{4}{\sqrt{x^2+16}}\right| - \ln\left|1-\frac{4}{\sqrt{x^2+16}}\right|\right] + C$$

<u>Ex</u> $\int_0^1 x(1+x^2)^{20}\,dx$ substitution

be careful w/ limits of integration.

let $u=1+x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$x=0$ — $u=1$

$x=1$ — $u=2$

$$\int_1^2 \frac{1}{2}u^{20}\,du = \frac{1}{2}\cdot\frac{1}{21}u^{21}\Big|_1^2$$

$$= \frac{1}{42}\left[2^{21}-2\right]$$

<u>Ex</u> $\int_1^e (\ln x)^2\,dx$ Integration by parts.

let $u=(\ln x)^2$ $dv = dx$

$du = \frac{2\ln(x)}{x}\,dx$ $v = x$

$$= x(\ln(x))^2\Big|_1^e - 2\int_1^e \ln(x)\,dx$$

let $u=\ln(x)$ $dv=dx$

$du = \frac{1}{x}\,dx$ $v=x$

$$= x(\ln(x))^2\Big|_1^e - 2\left[x\ln(x)\Big|_1^e - \int_1^e dx\right]$$

$$= x(\ln(x))^2\Big|_1^e - 2x\ln(x)\Big|_1^e + 2x\Big|_1^e$$

note $\ln(1)=0$

$$= e(\ln(e))^2 - 2e\ln(e) + 2e - 2$$

$$= e - 2e + 2e - 1 = e - 2$$

Ex $\int (2x+1)e^{x^2}e^x\,dx \underset{\text{rewrite}}{=} \int (2x+1)\,e^{x^2+x}\,dx$

this is a substitution integral

let $u = x^2+x$

$du = 2x+1\,dx$

$dx = \frac{du}{2x+1}$

$$= \int e^u\,du = e^u + C = e^{x^2+x} + C$$

Ex $\int \cos(2\sin(x))\cos x\,dx \underset{\text{substitution.}}{=} \int \cos(2u)\,du$

let $u = \sin(x)$

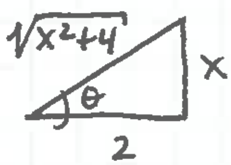
$du = \cos(x)\,dx$

$dx = \frac{du}{\cos(x)}$

$= \frac{1}{2}\sin(2u) + C$

$= \frac{1}{2}\sin(2\sin(x)) + C$

Ex $\int \frac{5x+6}{x^2+4}\,dx$ trigonometric substitution. $= \int [5[2\tan\theta]+6]\cdot\frac{1}{4}\cos^2\theta\cdot\frac{2d\theta}{\cos^2\theta}$



$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = \frac{2d\theta}{\cos^2\theta}$

$\cos\theta = \frac{2}{\sqrt{x^2+4}}$

$= \frac{1}{2}\int 10\tan\theta + 6\;d\theta$

$= \int 5\tan\theta + 3\;d\theta$

$= \int 5\frac{\sin\theta}{\cos\theta}\,d\theta + 3\theta + C$

let $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$d\theta = \frac{-du}{\sin\theta}$

$$= -5\int \frac{1}{u}\,du + 3\theta + C = -5\ln|u| + 3\theta + C = -5\ln|\cos\theta| + 3\theta + C$$

$$= -5\ln\left|\frac{2}{\sqrt{x^2+4}}\right| + 3\arctan\left(\frac{x}{2}\right) + C$$