[redacted] Day 11

Calculus

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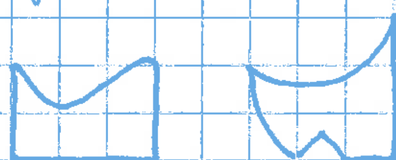
Today we start applications!

8.1 Area and Volume by integration

- some area and volume equations we have memorized:

$$A_{\circ} = \pi r^2 \qquad A_{\square} = bh \qquad A_{\triangle} = \tfrac{1}{2}bh$$

How would you find the area of:

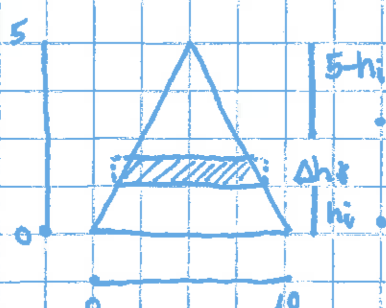


We can use <u>integration</u>.

<u>Ex</u> Lets start with one we know:

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(10)(5) = 25$$

Now calculate using integration.



- Imagine taking horizontal slices
- height is some small ammount Δh_i
- call the width w_i for slice i

like covering the triangle with strips of paper.

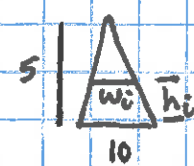
Area $= w_i \Delta h_i \rightarrow$ need to get everything in terms of h

The width depends on the height!

when $h_i = 0$ $w = 10$
when $h_i = 5$ $w = 0$ } linear

use ratios $\frac{w_i}{10} = \frac{5-h_i}{5} \longrightarrow w_i = \frac{10}{5}(5-h_i) = 2(5-h_i)$

the share the same top angle!



similar triangles.

so Area of Strip $= 2(5-h_i)\Delta h$

<u>Add them all up</u> Area $\approx \sum_{i=1}^{n} 2(5-h_i)\Delta h$

How do we make this more accurate? $\lim_{n\to\infty} \sum_{i=1}^{n} 2(5-h_i)\Delta h = \int_0^5 2(5-h)\,dh = $ Area

this gives the exact area

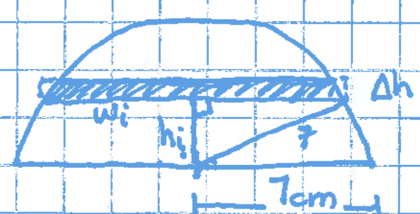
$$\text{Area} = 2\left[5h - \tfrac{1}{2}h^2\right]\Big|_0^5 = 2\left[5(5) - \tfrac{1}{2}(5)^2\right] - 0$$

$$= 2\left[25 - 25/2\right] = 50 - 25 = 25 \checkmark$$

same result

We start by trying shapes we know - so we can check our answers!

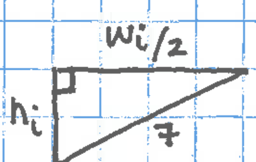
<u>Ex</u> Find the area of a semicircle of radius 7cm



1. Draw the picture with one slice
 Label the diagram

2. Area of the Slice: Area $= w_i \Delta h$

3. Get w_i in terms of h → need only one variable!



$a^2 + b^2 = c^2 \longrightarrow h_i^2 + b^2 = 49$
$b = w_i/2$

so $h_i^2 + \left(\frac{w_i}{2}\right)^2 = 49$ $w_i = \pm 2\sqrt{49 - h_i^2}$

choose + because can't have − width.

$A_i = 2\sqrt{49 - h_i^2}\,\Delta h.$

plug into area →

4. Add up all the pieces:

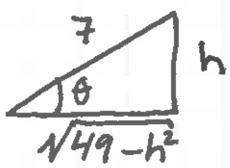
$$A \sim \sum_{i=1}^{n} w_i \Delta h = \sum_{i=1}^{n} 2\sqrt{49-h_i^2}\,\Delta h$$ then take the limit to make an integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\sqrt{49-h_i^2}\,\Delta h = \int_0^7 2\sqrt{49-h^2}\,dh$$

limits must go across all h!

5. Integrate to get final answer:

$$2\int_0^7 \sqrt{49-h^2}\,dh$$



$\sin\theta = h/7$

$h = 7\sin\theta$
$dh = 7\cos\theta\,d\theta$
$7\cos\theta = \sqrt{49-h^2}$

$$2\int_0^{\pi/2} 49\cos^2\theta\,d\theta = 49\int_0^{\pi/2} (1+\cos 2\theta)\,d\theta$$

$$= 49\left[\theta + \tfrac{1}{2}\sin 2\theta\right]\Big|_0^{\pi/2} = 49\pi/2$$

ALSO:

when $h=0$ $\quad 0 = 7\sin\theta \quad \theta = 0$ when $\sin\theta = 0$

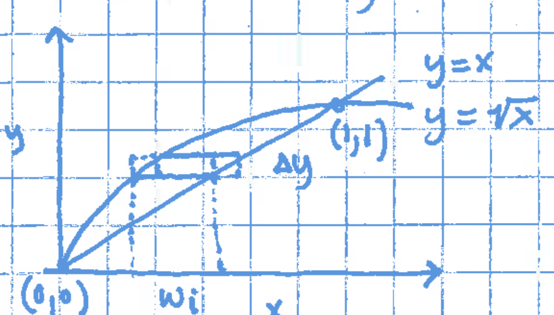
$h = 7 \quad 7 = 7\sin\theta \quad \theta = \pi/2$ when $\sin\theta = 1$

This matches what we have memorized $A = \pi r^2$

$$A = \tfrac{1}{2}(\pi r^2) = 49\pi/2 \ ✓$$

Now lets start dealing with areas we have no eqn for...
Same method different shape.

EX



Find the area between the two curves.

1. Draw and label graph!

2. Find intersection points!

$x = \sqrt{x} \qquad x^2 = x$

$x^2 - x = 0 \quad x(x-1) = 0$

$x=0 \quad x=1$

3. Find the area of a small slice.

$$\text{Area} = w_i \, \Delta y \quad (\text{base} \cdot \text{height})$$

4. Get width in terms of height variable.

$$w_i = y_i - y_i^2$$

difference between the line and curve!

"from where to where" in y.

$$\text{Area} = (y_i - y_i^2)\,\Delta y$$

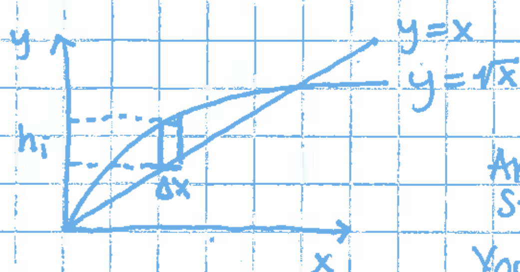
5. Sum up all area $\longrightarrow$ Integration

$$\lim_{n\to\infty} \sum_{i=1}^{n} y_i - y_i^2 \,\Delta y = \int_0^1 y - y^2 \, dy$$

evaluate integral and we're done!

$$\int_0^1 y - y^2 \, dy = \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = 1/6.$$

We could do this the other way around!



YOU TRY

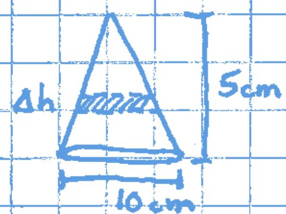
$$\text{Area of strip} = h_i \, \Delta x$$

You try to find the integral here — write down all your steps!

$$\int_0^1 \sqrt{x} - x \, dx = 1/6 \text{ same answer.}$$

We can use the same idea on a volume!

Find the volume of a cone w/ base diameter 10cm and height 5cm.



1. Draw and Label Picture

2. Consider a narrow horizontal slice:

it would be a disk:

$Area = \pi r_i^2 \Delta h$ the radius changes depending on height.

3. Get radius in terms of height...

again use ratios because we are dealing w/ triangles w/ common top angle

$\frac{2r_i}{10} = \frac{5-h_i}{5}$ so $2r_i = 2(5-h_i)$

$\underset{\text{of small slice}}{Area} = \pi(5-h_i)^2 \Delta h$

4. Add up all the tiny volumes: $A \sim \sum_{i=1}^{\infty} \pi(5-h_i)^2 \Delta h$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} 4\pi(5-h_i)^2 \Delta h = \int_0^5 \pi(5-h)^2 dh$

5. Solve Integral to get area:

$\pi \int_0^5 25 - 10h + h^2 dh = \pi\left[25h - 5h^2 + \frac{1}{3}h^3\right]\Big|_0^5$

$= \pi\left[125 - 125 + \frac{1}{3}125\right] = \frac{125}{3}\pi \ cm^3$

1. Draw and label graph
   - Include intersection points and important measurements base, height, radius.
2. Consider a small slice
   horizontal or vertical - SIMPLE.
   Find the AREA or VOLUME of the slice.
3. Rewrite AREA or VOLUME in terms of one variable.
   ex. If slices have Δx - then all variables should be x's.
4. Add up the slices - Reimann Sum - Take the limit to get an integral.
   - Limits of integration max and min possible values for your variable
5. Integrate to get your solution.