Day 12 – Numerical Integration.

We are starting to run into integrals that we cannot solve by hand.

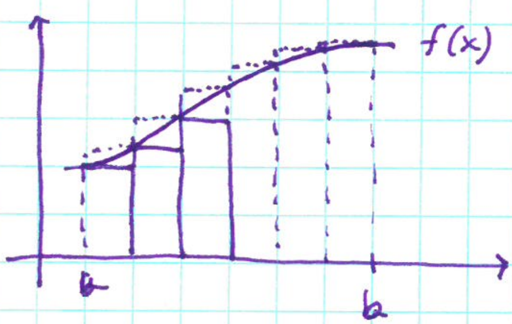
Ex. Arc Length: $\int_a^b \sqrt{1+f'(x)^2}\,dx$

this leads to complicated integrals!

We want to start talking about approximation... when we can't get and exact, analytical, solution we:

- Ask if there is a way to approximate the solution.
- Try to understand how much error is in our approximation.

RIGHT AND LEFT HAND RULES:



LEFT(n) → using n-divisions or n-data points... taking the height of box from the left.

$LEFT(N) = \Delta x\,(f(a) + f(c_2) + \cdots f(c_n))$

RIGHT(n) → n-data points... taking data from the right.

$RIGHT(N) = \Delta x\,(f(c_1) + f(c_2) + \cdots f(b))$

You should have seen these before..

What is the error here?

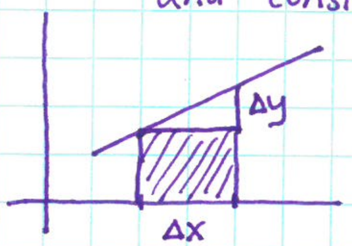
It depends on how steep f(x) is.
It depends on how many sub divisions.
It depends on how wide the range (a,b) is.

$|E| \le \dfrac{K(b-a)^2}{2n}$ where $K = \max(|f'(x)|)$ on $[a,b)$ → maximum absolute value of $f'(x)$

Increasing vs. Decreasing functions

Show this is true:

Assume $f(x)$ is linear with slope K.
and consider one slice:



Error is the triangle!

slope = $\frac{\text{rise}}{\text{run}}$

$\Delta x = \frac{(b-a)}{n}$ and $\Delta y = K\Delta x$

$$\underset{\text{one slice}}{\text{Error}} = \text{Area of Triangle} = \frac{1}{2}\Delta x \Delta y = \frac{K}{2}\Delta x^2 = \frac{K(b-a)^2}{2n^2}$$

$$\underset{\text{Total}}{\text{Error}} = \underbrace{n}_{\#\text{ of slices}} \cdot \underbrace{\frac{K(b-a)^2}{2n^2}}_{\text{error each slice}} = \frac{K(b-a)^2}{2n}$$

Linear is actually a worst case so for an error bound

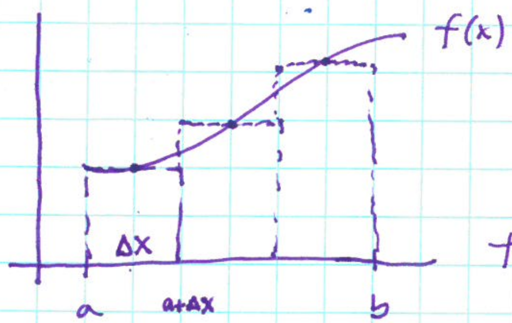
$$|E| \leq \frac{K(b-a)^2}{2n}$$

NOTE. We call these FIRST ORDER METHODS ... because the error decrease is proportional to $\frac{1}{n}$ so if we decrease Δx by half or double n we decrease Error by $\frac{1}{2}$.

Well now we say: wouldn't it be nice if we had "higher order" approximations? For the same # of points n we get less error! In otherwords, lets get a better approximation with less work! LAZY!

MIDPOINT and TRAPEZOIDAL RULES:

MID(n) — using n-data points construct the approximation using data from the middle of the slices.

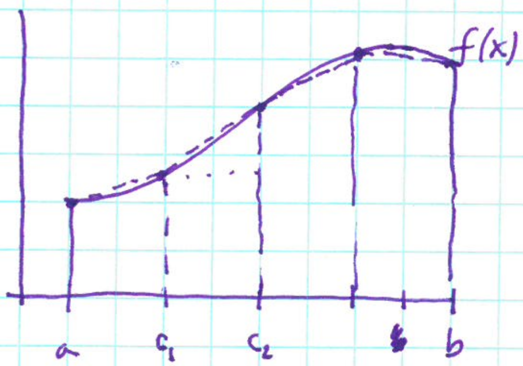


if our rectangles are at the points $a,\ a+\Delta x,\ a+2\Delta x,\ \dots,\ b$

then we take the height data from $a+\Delta x-\frac{1}{2}\Delta x,\ a+2\Delta x-\frac{1}{2}\Delta x,\ \dots,\ b-\frac{1}{2}\Delta x$

$$MID(n)=\left(f(c_1)+f(c_2)+\cdots+f(c_n)\right)\Delta x \qquad x_j=a+(j-\tfrac{1}{2})\Delta x$$

TRAP(n) — rather than approximating using n-rectangles, use n-trapezoids.



Top of trapezoid is a secant line.

$$TRAP(N)=\frac{1}{2}\Delta x\left(f(a)+2f(c_1)+2f(c_2)+\dots f(b)\right)$$

$Area_1=f(a)\Delta x+\frac{1}{2}(f(c_1)-f(a))\Delta x$ .... $AREA_2=f(c_1)\Delta x+\frac{1}{2}(f(c_2)-f(c_1))\Delta x$

add these up and factor out $\frac{1}{2}\Delta x$

The error of these approximations now depends on
- curvature of $f(x)$
- number of divisions N
- the distance (a,b)

$$\text{Error}(MID_N)\le\frac{K_2(b-a)^3}{24N^2} \qquad \text{Error}(TRAP_N)\le\frac{K_2(b-a)^3}{12N^2}$$

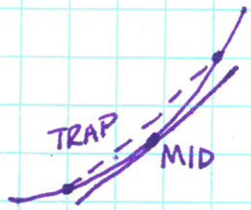
where K_2 is the $\max|f''(x)|$ for $x\in[a,b]$

What would we expect if $f(x)$ is linear? ZERO ERROR! does this make sense?
Concave up vs. Concave Down.

These are called second order methods. double N ⇒ Error divide by 4

SIMPSONS RULE ... we can actually do better than second order.

Lets look at TRAP and MID on one slice:



TRAP over estimates

MID — actually corresponds to a tangent line ... because the "over" "under" cancels out.

The idea ... the true soln lies between the TRAP and MID ... why not take an average of these?

$$SIMP(n) = \frac{1}{3} TRAP(N/2) + \frac{2}{3} MID(N/2)$$

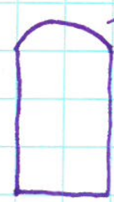
the weights are based on the fact that MID(n) is ½ the error of TRAP(n).

$$\text{Error}(SIMP_N) \leq \frac{K_4 (b-a)^5}{180 N^4}$$

fourth order method!

$$K_4 = \max\{|f^4(x)|\} \quad x \in [a,b].$$

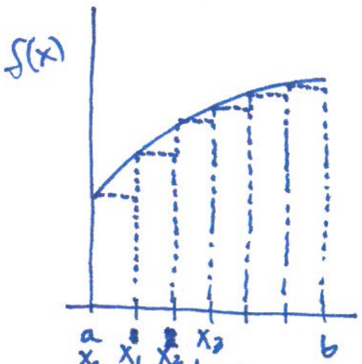
This is a parabolic approximation.



Top of "rectangles" is now approximated by a parabola.

Approximating Definite Integrals: $\int_a^b f(x)\,dx$

Left Hand Sum



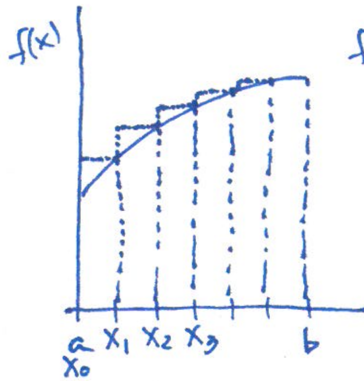
counts up Area of rectangles.

gets data from left hand side.

$f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$

LEFT(n) = left hand sum with n divisions

Right Hand Sum



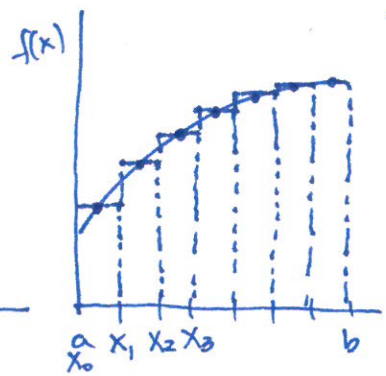
counts up area of rectangles

gets data from right hand side

$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$

RIGHT(n) = right hand sum with n divisions

Midpoint Rule



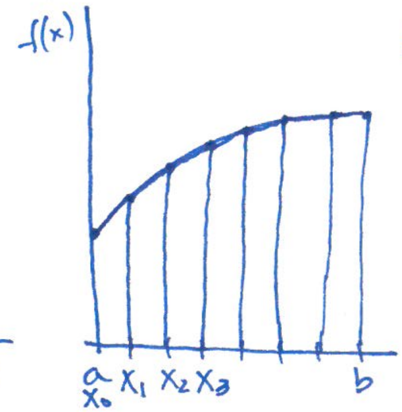
counts up area of rectangles

gets data from middle

$f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x + \dots$

MID(n) = midpoint sum with n divisions

Trapezoid Rule



counts up area of trapezoids

averages left and right data

$\frac{f(x_0)+f(x_1)}{2}\Delta x$

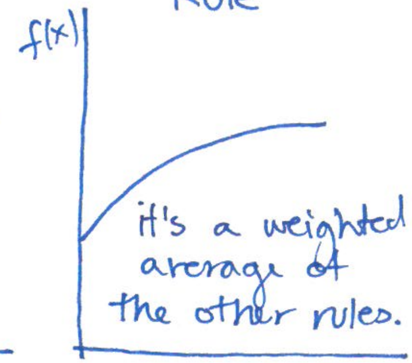
$+ \frac{f(x_1)+f(x_2)}{2}\Delta x + \dots$

OR

Average of left and right hand sums.

$TRAP(n) = \frac{LEFT(n) + RIGHT(n)}{2}$

Simpsons Rule



$SIMP(n) = \frac{2\,MID(n) + TRAP(n)}{3}$

Remember $\boxed{\Delta x = \frac{b-a}{n}}$

Numerical Integration.

Now we need to talk about ERROR $\longrightarrow$

Error in Approximating $\int_a^b f(x)\,dx.$

$$\boxed{\text{Error} = \text{Actual Value} - \text{Approximate Value}}$$

Left and Right Hand Sums.

OVER OR UNDERESTIMATE?

if $f(x)$ is increasing on $[a,b]$

$$\text{LEFT}(n) \le \int_a^b f(x)\,dx \le \text{RIGHT}(n)$$

if $f(x)$ is decreasing on $[a,b]$

$$\text{RIGHT}(n) \le \int_a^b f(x)\,dx \le \text{LEFT}(n)$$

MIDPOINT and TRAPEZOIDAL rules

OVER OR UNDERESTIMATE?

if $f(x)$ is concave down on $[a,b]$

$$\text{TRAP}(n) \le \int_a^b f(x)\,dx \le \text{MID}(n)$$

if $f(x)$ is concave up on $[a,b]$

$$\text{MID}(n) \le \int_a^b f(x)\,dx \le \text{TRAP}(n)$$

Error Bounds

Midpoint Rule

$M =$ maximum of $|f''(x)|$ in $[a,b]$.

absolute value of error $= |e_M| \le \dfrac{M(b-a)^3}{24n^2}$

Trapezoid Rule

$M =$ maximum of $|f''(x)|$ in $[a,b]$

absolute value of error $= |e_T| \le \dfrac{M(b-a)^3}{12n^2}$

SIMPSONS RULE

Error Bounds

$M =$ maximum of $|f^4(x)|$ in $[a,b]$

fourth derivative

absolute value of error $= |e_S| \le \dfrac{M(b-a)^5}{180n^4}$