# Day 18 – Numerical Integration.

Recall: $\Delta x = \frac{(b-a)}{N}$ ← width of "boxes"; $N$ ← # of "boxes"

**First Order**

$$L_N = \Delta x\left(f(a) + f(c_2) + \cdots + f(c_n)\right) \qquad c_i = a + (1-i)\Delta x$$

$$R_N = \Delta x\left(f(c_1) + f(c_2) + \cdots + f(c_{n-1}) + f(b)\right) \qquad c_i = a + i\Delta x$$

**Second Order**

$$M_N = \Delta x\left(f(c_1) + f(c_2) + \cdots + f(c_n)\right) \qquad c_i = a + (i - 1/2)\Delta x$$

$$T_N = \tfrac{1}{2}\Delta x\left(f(a) + 2f(c_1) + 2f(c_2) + \cdots 2f(c_{n-1}) + f(b)\right) \qquad c_i = a + i\Delta x$$

**Fourth Order**

$$S_N = \frac{1}{3}T_{N/2} + \frac{2}{3}M_{N/2}$$

Error:

$$|E_{LR}| \le \frac{K_1(b-a)^2}{2N}$$

$$K_1 = \max\{|f'(x)|\} \quad x \in [a,b]$$

$$|E_M| \le \frac{K_2(b-a)^3}{24N^2} \qquad |E_T| \le \frac{K_2(b-a)^3}{12N^2}$$

$$K_2 = \max\{|f''(x)|\} \quad x \in [a,b]$$

$$|E_S| \le \frac{K_4(b-a)^5}{180N^4}$$

$$K_4 = \max\{|f^{(4)}(x)|\} \quad x \in [a,b]$$

How do we use these?

Ex. Estimate $\int_0^4 \sqrt{x}\,dx$ using numerical integration.

First... how many points, $N$, do you need to estimate the answer to within an error tolerance of $|E| < 10^{-4}$?

Second... will you over or under estimate or can you not tell?

## Error for LEFT and RIGHT

1. Find $K_1$: $f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

then for $x$ between 2 and 4 what is the biggest this would be?

$f'(x) = \frac{1}{2\sqrt{x}}$ so $|f'(2)| > |f'(4)|$ and it doesn't do anything weird inbetween. graph the function if you need to.

$K_1 = \frac{1}{2\sqrt{2}} < \frac{1}{2}$ its okay to choose a slightly larger $K_1$ ...

2. plug into error formula:

$$|E_{L,R}| < \frac{(2)^2}{4N} = \frac{1}{4N}$$

3. consider error tollerance we want $|E| < 10^{-4}$

$$\text{so } \frac{1}{N} < 10^{-4} \Rightarrow N > \frac{1}{10^{-4}} = 10^4$$

I would need 10,000 points to get the desired accuracy.

Because $f(x) = \sqrt{x}$ is increasing between $2 < x < 4$ I know that $L_N < \int_2^4 \sqrt{x}\,dx < R_N$

What if this integral represented the total allowable safe dose of a drug to adminsster to a patient ... which estimate would you choose?

## YOU TRY ERROR MID, TRAP, and SIMP:

$K_2 < \frac{1}{8}$ $f(x)$ is concave down ... $T_N$ is under and $M_N$ over.

$$|E_M| < \frac{(2)^3}{8 \cdot 24 N^2} = \frac{1}{24N^2} < 10^{-4} \qquad |E_T| < \frac{(2)^3}{8 \cdot 12 N^2} = \frac{1}{12N^2} < 10^{-4}$$

$\frac{1}{24} 10^4 < N^2$ $N > \sqrt{\frac{10^4}{24}} = 20.41$ $N = 21$

$N > \sqrt{\frac{10^4}{12}} = 28.87$ $N = 29$

$K_4 < \frac{15}{48}$ $\quad |E_S| < \frac{15(2)^5}{48 \cdot 180 \cdot N^4} = \frac{15 \cdot 32}{48 \cdot 180 N^4} = \frac{480}{8640} \frac{1}{N^4} < 10^{-3}$

So $N^4 > \frac{480}{8640} \cdot 10^4 = 555.55$

Don't know over or under!

$N > (555.55)^{1/4} = 4.85$

$N = 5$ will work.

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Usually we use computers to do these calculations!
TODAY we will use a spreadsheet ... but you can write code in any language to do this.

First... what is the exact answer?

$$\int_2^4 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_2^4 = \frac{2}{3}\left[\sqrt{4^3} - \sqrt{2^3}\right]$$

$$= \frac{2}{3}\left[8 - 2\sqrt{2}\right] \sim 3.448$$

STEPS ... PSUEDO CODE ...

1. input a, b, and N
2. calculate $\Delta X$
3. build data table MAIN — X | f(x)
4. build data table MID — X* | f(x*)
5. input $L_N$, $R_N$, $M_N$, $T_N$ and then $S_N$

} Check error tolerance by hand! or at least calculate derivatives by hand

SPREADSHEET HINTS:
variable vs. static references: I can "drag" a cell to change the input

= C4 ↓ drag

spreadsheet auto changes the #

=5
=C6 etc

= $C$4 drag ↓

=$C$4

static reference
the # does not change
ONLY LOOKS ONE PLACE.

## Important Functions

| | |
|---|---|
| =SQRT(x) | $\sqrt{x}$ |
| =SIN(x) | $\sin(x)$ |
| =EXP(x) | $e^x$ |
| =X^2 | $X^2$ |
| =SUM(A1:A10) | adds entries in A1, A2, A3 ..., A10 |
| =C2 | references data in cell C2 |
| =$C$2 | static reference to data in cell C2 |

If you have questions google is a good place to start!