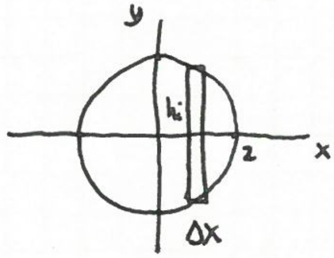
# 8.3 Area: Integration in Polar Coordinates:

First: Set up the integral that gives the area of the circle with radius 2, given $x^2+y^2=4$




Slice either vertically or horizontally.


$A_i = h_i\,\Delta x$

depends on x

Use the equation:

$x^2+y^2=4$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

so

$h_i = y_{TOP} - y_{BOT}$

$h_i = \sqrt{4-x^2} - (-\sqrt{4-x^2})$

$h_i = 2\sqrt{4-x^2}$

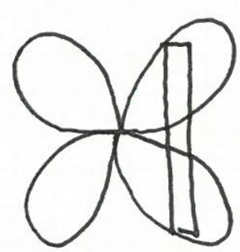
$A_i = 2\sqrt{4-x^2}\,\Delta x$

$$A_{TOTAL} = 2\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

Solve by trig substitution.

Problem – this is complicated because we are representing a circular area with a square grid – rectangles.

How would we find the area of:

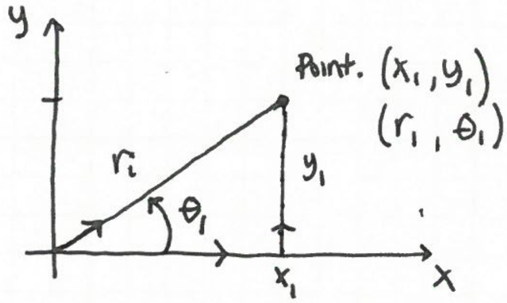


umm... these slices would be a mess!

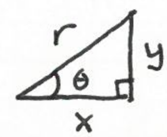
Polar Coordinates are our FRIEND!

Review: Polar Coordinates and Graphs.

Instead of $(x, y)$ we can use $(r, \theta)$ and get to the same point.




How do we relate $(x, y)$ and $(r, \theta)$




triangle!!

$\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

<u>so</u> $x = r\cos\theta$ and $y = r\sin\theta$

<u>also</u> $x^2 + y^2 = r^2$.

Worksheet. As a Class <u>Do</u> #1-5 and <u>Stop</u>.

Graphing in polar: TABLE OF VALUES at important points.

<u>Ex</u> $r = 3\sin(2\theta)$

Choose points in $\theta$ where $\sin(2\theta) = 0, 1, 0, -1,$ etc.

This happens when $2\theta = 0, \pi/2, \pi, 3\pi/2, \ldots$ so choose $\theta = 0, \pi/4, \pi/2, 3\pi/4, \ldots$

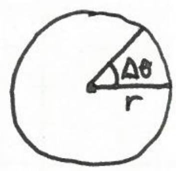
| $\theta$ | $r = 3\sin(2\theta)$ |
|---|---|
| 0 | 0 → in between $\theta = \pi/6$ $r = 2.598$ |
| $\pi/4$ | 3 |
| $\pi/2$ | 0 |
| $3\pi/4$ | $-3$ |
| $\pi$ | 0 |
| ⋮ | |

notice the pattern? If needed try some in between points

Then Graph. Remember $r =$ negative find the angle then graph "behind" or "backward"

Now – lets develop <u>Integration in Polar</u> USE RADIANS.

Think about a circle

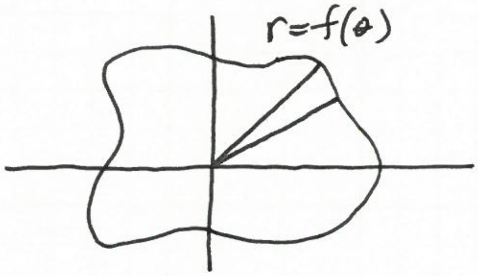



A slice is just a part of the circle.

Whole Circle : $\pi r^2$

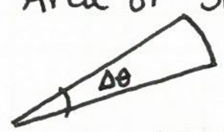
Slice of Circle : $\frac{\Delta\theta}{2\pi}\pi r^2 = \frac{1}{2}r^2\Delta\theta.$

More Complicated Shapes :




Our slice is still a sector or piece of pizza.

Area of Slice:




Same

$A_i = \frac{1}{2}r_i^2\Delta\theta$

<u>Now</u> $r_i$ depends on $\theta$ and this is given by.

$r = f(\theta)$

<u>So</u> $A_i = \frac{1}{2}(f(\theta))^2\Delta\theta$

$$A_T = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}f(\theta)^2\Delta\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta$$

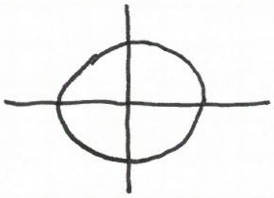
$\alpha$ is the minimum angle $\beta$ is the maximum angle.

Often these are hard to find. Really think about what $\theta$ needs to be to sweep out your shape

$$\text{AREA} = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta.$$

<u>Ex</u> Find the area of the circle $x^2+y^2=4$

<u>IN POLAR</u> $r^2=4$ or $r=2$



to sweep out the whole circle $0 \le \theta \le 2\pi$

so $\text{Area} = \frac{1}{2}\int_0^{2\pi} (2)^2 \, d\theta = 2\theta \Big|_0^{2\pi} = 4\pi$ ✓

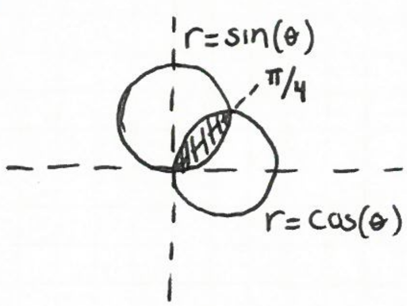
Way easier!

<u>WORKSHEET</u> — lets do #2 as a class.

Then you try #3.

<u>Ex</u> A more complicated case. Find the area where the curves $r=\sin(\theta)$ and $r=\cos(\theta)$ overlap.

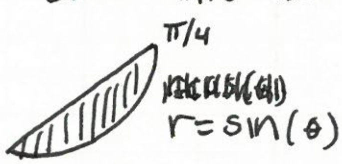
DRAW




Need intersection points:

$\sin\theta = \cos\theta$ when $\theta = \pi/4$ $\quad \sin(\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

Break into two parts:

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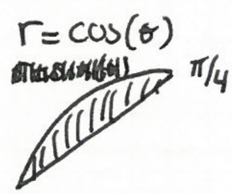


$$A_T = \frac{1}{2}\int_0^{\pi/4} \sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta$$