TODAY

Complicated Examples... adding up more interesting quantities.

<u>Ex</u> Find the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

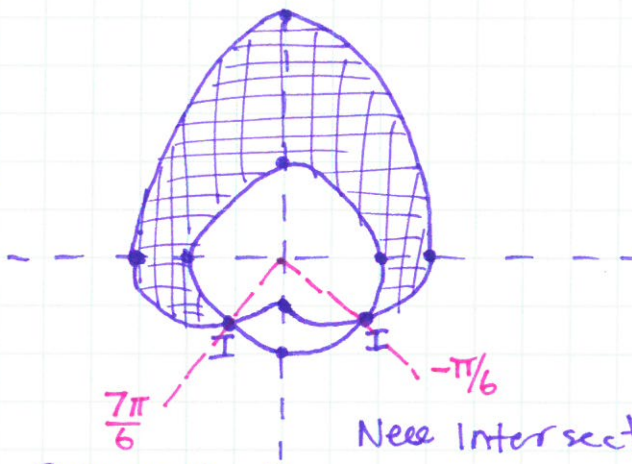
$$\frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\,d\theta$$

Draw the curves:

choose: $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$

θ	$r = 3 + 2\sin\theta$
0	3
$\pi/2$	5
π	3
$3\pi/2$	1
2π	3

$r=2$ circle



Need intersection points.

Where is $3 + 2\sin\theta = 2$

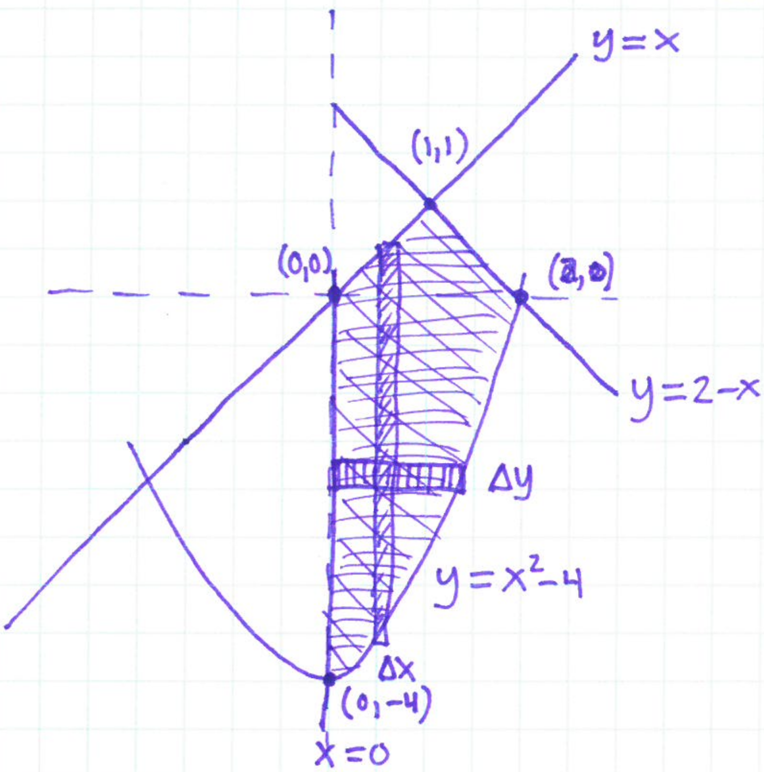
$2\sin\theta = -1$

$\sin\theta = -1/2 \qquad \theta = -\pi/6$ and $7\pi/6$

Need the area between the two curves... OUTER AREA $-$ INNER AREA

$$\int_{-\pi/6}^{7\pi/6} \frac{1}{2}(3 + 2\sin\theta)^2\,d\theta - \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(2)^2\,d\theta$$

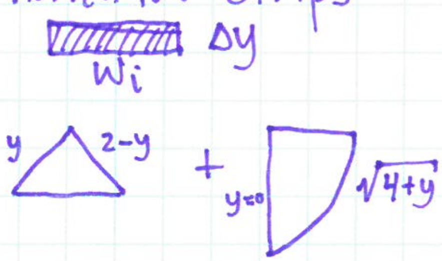
Ex Find the area bounded by the curves:
$y=x$, $y=2-x$, $y=x^2-4$ and $x\geq 0$.



This region is not simple in any direction!

Choose to integrate either direction...

Horizontal strips

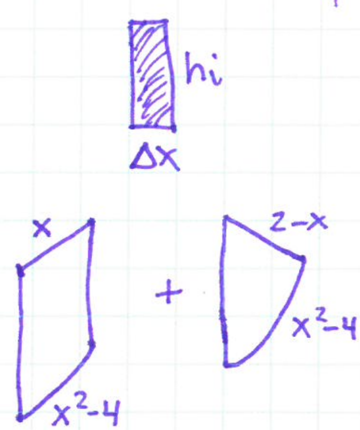


$A_i = w_i\,\Delta y \quad + \quad A_i = w_i\,\Delta y$

$w_i = 2-y-y \quad + \quad w_i = \sqrt{4+y}$

$$A_T = \int_0^1 2-2y\,dy + \int_{-4}^0 \sqrt{4+y}\,dy$$

Horizontal strips



$A_i = h_i\,\Delta x$

$h_i = x - x^2 + 4$

$A_i = h_i\,\Delta x$

$h_i = 2 - x - x^2 + 4$

$$A_T = \int_0^1 x - x^2 + 4\,dx + \int_1^2 2 - x - x^2 + 4\,dx$$

If we can do Volume by integration we can find other quantities!

DENSITY or MASS

Recall If I have an object with Area = $4m^2$ and density = $2\ kg/m^2$

How do I find mass?

DENSITY x AREA = $4m^2 \times 2\ kg/m^2 = 8\ kg$.

Same idea w/ Integration! Two cases:

(A) Constant Density: if the density is a constant then we can find the LENGTH, AREA, or VOLUME first Then multiply by DENSITY.

Ex Density $\rho(x) = 3\ kg/m^2$ in the shape we solved before: $y=x$ and $y=\sqrt{x}$. (m^2)

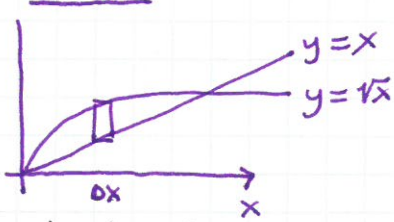
$$\text{Area} = \int_0^1 \sqrt{x} - x\,dx = \frac{1}{6}\ m^2$$

so Mass $= 3 \times \frac{1}{6} = \frac{1}{2}$ kg.

(B) Non-Constant Density: if density is not constant we must be more careful!

Ex Density $\rho(x) = 3x\ kg/m^2$ in the shape we solved before: $y=x$ $y=\sqrt{x}$

Picture



each slice has different density.

now we need to add up all the masses!

Consider a slice

$\rho(x_i) = 3x_i$ for the slice

$$\text{Mass}_i = \text{density}_i \times \text{area}_i = \underbrace{3x_i}_{\text{density}}\ \underbrace{h_i \Delta x}_{\text{area}}$$

In terms of x $\longrightarrow$ $\text{Mass}_i = 3x_i(\sqrt{x_i} - x)\Delta x$

Add up the slices: $\sum_{i=1}^{n} 3x_i(\sqrt{x_i} - x_i)\Delta x$

$$\lim_{n\to\infty} \sum_{i=1}^{n} 3x_i(\sqrt{x_i} - x_i)\Delta x = \int_0^1 3x(\sqrt{x} - x)\,dx$$

NOTE: The only difference is that we multiply the inside by the density!

IN GENERAL:

Mass Linear 1-D : $\int_a^b \rho(x)\,dx$

Mass 2-D : $\int_a^b \rho(x) \cdot \text{length}\,dx$

Mass 3-D : $\int_a^b \rho(x) \cdot \text{Area cross-section}\,dx$

Ex Find the total mass of a 2m rod of linear density $\rho(x) = 1 + x(2-x)$ kg/m

Picture of rod:

A slice has length Δx

$\text{mass}_i = \rho(x_i)\Delta x$

This is linear density times length.

$$\text{TOTAL MASS} = \sum_{i=1}^{n} 1 + x_i(2-x_i)\Delta x = \int_0^2 1 + x(2-x)\,dx$$

how did I get these limits?

"Do the same thing you would normally do just under an integral sign" "add up the pieces"

We can use this idea of density in a variety of ways.

Population $\longrightarrow$ often the population density, # of people per square mile, depends on the distance from some population center.

$\rho = \rho(r)$ radial dependence

If we know $\rho(r)$ we can find total population.

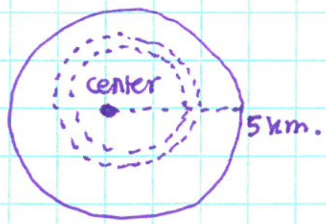
Ex Odzala National Park in the Republic of Congo has a high density of gorillas. Suppose that by empirical modeling we found

$$\rho(r) = 10(1+r^2)^{-2} \text{ gorillas per square km.}$$

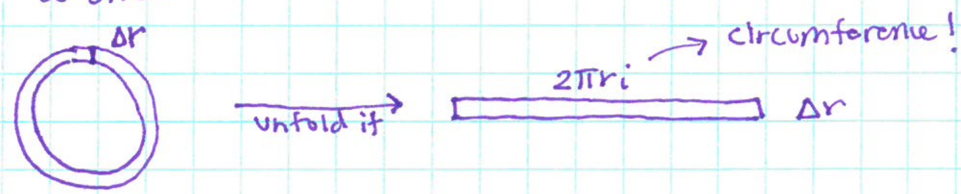
where r is the radial distance from a large grassy clearing w a source of food and water.

Find the # of gorillas within a 5 km radius.

Draw a Picture



Consider a slice — how do we slice this?



The population in the strip = area of strip km^2 $\times$ population density $\text{gorillas}/km^2$

$= 2\pi r_i \, \rho(r_i) \, \Delta r$

so $\text{Population} = \int 2\pi r \rho(r)\, dr \Rightarrow \int_0^5 2\pi r \cdot \frac{10}{(1+r^2)^2}\, dr \cong 60.22.$

we just need to plug in $\rho(r)$

and find limits