Day 19.

Goal Build a foundation in sequences and series...
- experience some fun math
- approximate functions or #'s using
  a sum of polynomials.

Today: Sequences ... lets learn about the basics before jumping straight into more complicated

A sequence is an ordered collection of numbers.

$$a_1, a_2, a_3, \ldots, a_n$$

each of the a_n's is a term in the sequence and n is the index. often they are defined by a discrete function $f(n)$.

Ex.

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$ we are dividing each successive number in half.

$f(n) = a_n = \frac{1}{2^n}$ $n \geq 0$

this is the general term.

Ex. What is the sequence? $a_n = 1 - \frac{1}{n}$ $n \geq 1$ $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

Ex. What is the general term? $0, -1, 2, -3, \ldots$

$a_n = (-1)^n n$ $n \geq 0$.

We also sometimes define sequences recursively:

$$a_1 = 1 \qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

$\uparrow$ initial or starting value

$\rightarrow$ generating recursion relation

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

OUR GOAL ... eventually we want to add up a sequence of numbers to represent functions or other numbers.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

$\rightarrow$ note when your calculator tells you π it is an approximation

In the past, before everyone had calculators, this was more obviously useful. <u>But</u> many disciplines use this ... approximation theory is hugely important in science.

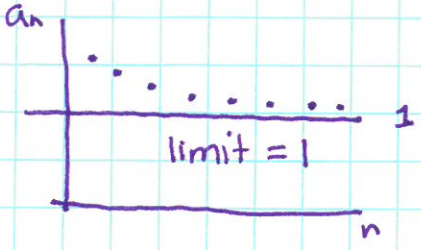
We are doing calculus with these things ... we need LIMITS AND CONVERGENCE.

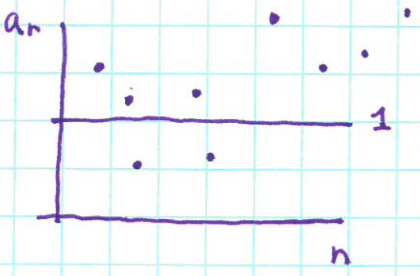
LIMIT OF A SEQUENCE.

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

similar to what we see with functions.

- if no limit exists we say a_n diverges
- if the terms increase with no bound a_n diverges to ∞.





Diverges... no limit

Ex $a_n = \dfrac{n+4}{n+1}$

what is the limit here?

$4, \frac{5}{2}, \frac{6}{3}, \frac{7}{4}, \ldots$ guesses?

$$\begin{array}{r l} & \;\;1 \\ n+1 & \overline{)\,n+4} \\ & \;\;\underline{n+1} \\ & \quad\;\; 3 \end{array} = 1 + \frac{3}{n+1} \quad \text{as } n \to \infty \quad a_n \to 1$$

We can do similar things when our sequence is defined by a function. In otherwords replace $a_n \to f(x)$

Ex. Find the limit $\displaystyle\lim_{n\to\infty} \frac{n + \ln(n)}{n^2} = \lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x + \ln(x)}{x^2}$$

$$\boxed{\begin{array}{c}\text{to evaluate this we need L'Hopitals Rule:} \\ \displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \\ \text{For limits at } \pm\infty.\end{array}}$$

$f(x) = x + \ln(x)$

$f'(x) = 1 + 1/x$

$g(x) = x^2$

$g'(x) = 2x$

$$\lim_{x\to\infty} \frac{1 + 1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x} + \frac{1}{2x^2} = 0$$

<u>RULES FOR LIMITS...</u> Assume $\displaystyle\lim_{n\to\infty} a_n = L$ and $\displaystyle\lim_{n\to\infty} b_n = M$

i $\displaystyle\lim_{n\to\infty} (a_n \pm b_n) = \lim_{a\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$

ii $\displaystyle\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = LM$

iii $\dfrac{\displaystyle\lim_{n\to\infty} a_n}{\displaystyle\lim_{n\to\infty} b_n} = \dfrac{L}{M}$ assuming $M \neq 0$

iv $\displaystyle\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ c = constant.

BOUNDED SEQUENCE

- Bounded from above: there is a number M such that $a_n \leq M$ for all n.
  M is the <u>upper bound</u>
- Bounded from below:
  $a_n \geq m$ for all n.
  m is the <u>lower bound</u>

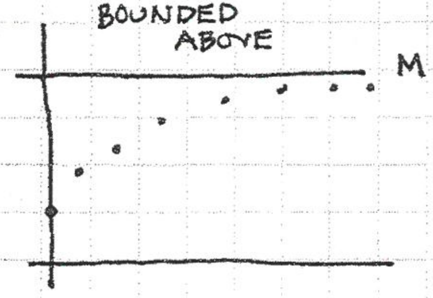
If $m \leq a_n \leq M$ it is bounded from above and below ... we call it a <u>bounded sequence</u> otherwise it is <u>unbounded</u>.

All convergent sequences are BOUNDED. However not all BOUNDED sequences converge.

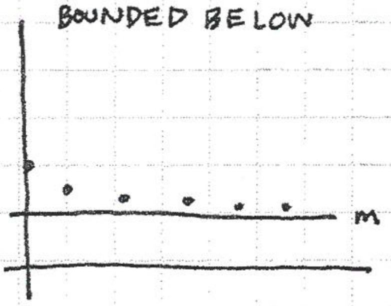
If a sequence is BOUNDED <u>and</u> MONOTONIC then it converges.

- $\{a_n\}$ is ONLY increasing $a_n < a_{n+1}$ for all n
- $\{a_n\}$ is ONLY decreasing $a_n > a_{n+1}$ for all n.

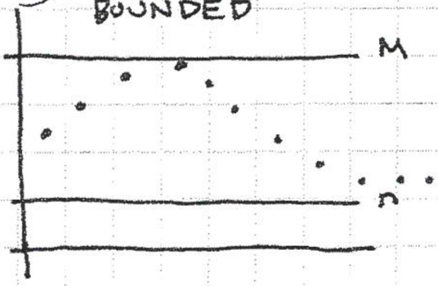
BOUNDED ABOVE



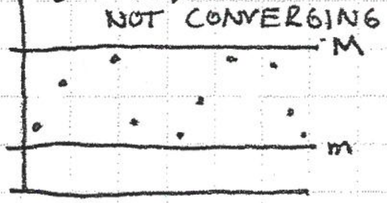
BOUNDED BELOW



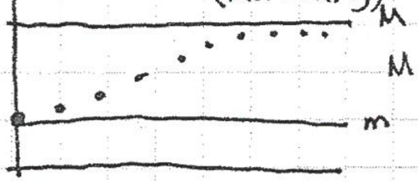
BOUNDED



BOUNDED NOT CONVERGING



BOUNDED MONOTONIC (increasing)



Must converge. !!

MATH 122 - In-Class Worksheet Day 19

A sequence is an ordered collection of numbers $a_1, a_2, a_3, \dots a_n$, where each of the a_n's are a term in the sequence with index n. We will use the following notation:

$$\{a_n\}_{n=1}^{\infty} = \{a_n\} = a_1, a_2, a_3, \dots$$

Often sequences are defined by a discrete function. For example,

$$f(n) = a_n = \frac{1}{2^n}$$

What is a_1? What is a_2? What is a_{10}?

It is also useful to be able to write down a general formula, given a sequence of numbers. Given the sequence $2, 4, 8, 16, 32, 64 \dots$ I would write down the formula $f(n) = 2^n$. Given the sequence $-1, 3, -5, 7, -9, 11 \dots$ I would write down the formula $f(n) = (-1)^n(2n-1)$. What does the $(-1)^n$ do? What does the $(2n-1)$ do?

Write down a general term for the sequence $1, 2, 4, 8, 16, 32 \dots$.

Sometimes we will define sequences using a recursive formula. For example,

$$a_n = a_{n-1} + 3$$

for $n > 1$ and $a_1 = 2$. What are the first six terms in this sequence?

A sequence an has limit L

$$\lim_{n\to\infty} a_n = L$$

if $a_n \to L$ as $n \to \infty$. This means that we can make the terms a_n arbitrarily close to L by making n sufficiently large. If

$$\lim_{n\to\infty} a_n$$

exists (and is equal to a real number), we say that the sequence $\{an\}$ converges (or is convergent). Otherwise, we say that the sequence diverges (or is divergent).

1. Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n}$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

2. Consider the sequence $\{a_n\}$ defined by

$$a_n = n + 2$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

3. Consider the sequence $\{a_n\}$ defined by

$$a_n = (-1)^n$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

4. Consider the sequence $\{a_n\}$ defined by

$$a_n = (\frac{1}{4})^n$$

Find $\lim_{n\to\infty} a_n$. Does the sequence $\{a_n\}$ converge or diverge?

5. Consider the sequence $\{a_n\}$ defined by

$$a_n = (p)^n$$

For what values of r is the sequence an convergent?

As a class we will write down some rules for limits and remember L'Hopitals rule: