Calculus 2
Day 20

Last Time: Sequences

* What is a Bounded Sequence?
* What is a Monotone Sequence?
* What does it mean if a sequence Converges? Diverges?

9.2 Geometric Series

When we add the terms in a sequence we get a Series:

* SEQUENCE $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ A list of numbers.

* SERIES $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ Adding up a list of numbers.

Lots of interesting questions arise ... If I add up all of the terms, infinitely many, what number do I get?

Very useful in systems where we notice patterns.

Ex Drug Dosage. How do doctors prescribe medicine?

You get an infection and are told to take antibiotics...

Take 4 250 mg doses each day. Every six hours.

But when you take the dose, your body starts breaking it down

After 6 hours 4% of the drug remains in the body

How much drug is in your system after the tenth dose?

Lets look at the process and hope for a pattern.

After the first dose: $Q_1 = 250$
second dose $Q_2 = 0.04(250) + 250$
third dose $Q_3 = 0.04[0.04(250)+250] + 250$
$= (0.04)^2(250) + (0.04)(250) + 250$

what is the pattern?

$$Q_n = (0.04)^{n-1}250 + (0.04)^{n-2}250 + \cdots + (0.04)250 + 250.$$

So after ten doses:

$$Q_{10} = (0.04)^9 250 + (0.04)^8 250 + \cdots + (0.04)250 + 250.$$

to find a value we can use a neat trick: Calculate $Q_{10} - (0.04)Q_{10}$.

Good Trick!

$$(0.04)Q_{10} = (0.04)^{10}250 + (0.04)^9 250 + \cdots + (0.04)^2 250 + (0.04)250$$

then $Q_{10} - (0.04)Q_{10} = -(0.04)^{10}250 + 250$

So

$$Q_{10} = \frac{250(1-(0.04)^{10})}{1-0.04} = 260.42 \text{ mg.}$$

The big question w/ drug dosing is: can we give the patient too much? In otherwords, if we keep them on the drug for 2 weeks what will happen?

- It looks like the concentration just keeps increasing.

Lets consider what hapens as $n \to \infty$

$$Q_n = \frac{250(1-(0.04)^n)}{1-0.04}$$

then $\lim_{n\to\infty} Q_n = \frac{250(1-0)}{1-0.04} = 260.42$

actually the drug wont increase... It has stabalized.

When we look at problems like this ... discrete doses or incremental procedures, we often end up with a GEOMETRIC SERIES.

A. Finite Geometric Series $\quad a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}$

B. Infinite Geometric Series $\quad a + ax + ax^2 + ax^3 + \cdots$

In our drug example
$a = 250$ and $x = 0.04$

What is the value of these when we add them up.

A. Consider the sum of the Finite Series.

$$S_n = a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}$$

Use the same trick.

consider $\quad xS_n = ax + ax^2 + ax^3 + ax^4 + \cdots + ax^n$

then subtract $\quad S_n - xS_n = a - ax^n$

THIS IS A PARTIAL SUM

and $\quad S_n = \dfrac{a - ax^n}{1-x} = \dfrac{a(1-x^n)}{1-x}$ $\quad$ assumes $x \neq 1$

where n is the # of terms.

The general form is nice so you can apply it to a wide range of problems!

B. What about the
Sum of the Infinite Series?
- lets add up an infinite # of terms!!

$$S_n = \frac{a(1-x^n)}{1-x}$$

consider $\displaystyle \lim_{n\to\infty} \frac{a(1-x^n)}{1-x}$ ... evaluate this limit.

It will only converge if $|x|<1$
otherwise x^n grows!

INFINITE SUM.

$$\lim_{n\to\infty} \frac{a(1-x^n)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x}$$
for $|x|<1$

For the case of $|x| \geq 1$ the series does not converge or it diverges.

Ex $|x|>1$ then x^n grows $\lim_{n\to\infty} s_n = \infty$ the series diverges.

$x=1$ then the series is $a+a+a+a+a\ldots$
the series diverges

$x=-1$ then the series is $a-a+a-a+a-\ldots$
oscilates between 0 and a
does not converge.

How do we use this?

Ex Future Value of a Company.

You own a company that makes cookies once a year for Valentines Day. Each year you make \$500 and you invest it in a continuously compounded bank account at 1%.

You and your heirs plan to run this company forever... what is the present value?

$$\text{Total Present Value} = 500 + 500e^{-0.01(1)} + 500e^{-0.01(2)} + 500e^{-0.01(3)} + \cdots$$

$$= 500 + 500(e^{-0.01}) + 500(e^{-0.01})^2 + 500(e^{-0.01})^3 + \cdots$$

this is geometric series w/ $a=500$ and $x=e^{-0.01}$

to add up all the value use the sum:

$$S = \frac{a}{1-x} = \frac{500}{1-e^{-0.01}} \cong \$50,250.42$$ total present value.

Ex Regular Deposits into a savings account.

Suppose you deposit \$1000 anually into an account earning 5% per year, compounded annually.

What is the ballance after the n^{th} deposit?

$$B_1 = 1000$$
$$B_2 = B_1(1.05) + 1000 = 1000(1.05) + 1000$$
$$B_3 = B_2(1.05) + 1000 = 1000(1.05)^2 + 1000(1.05) + 1000$$

this is a geometric series $a = 1000$, $x = 1.05$

$$B_n = \frac{1000(1-(1.05)^n)}{1-1.05} = \frac{1000((1.05)^n - 1)}{1.05 - 1}$$

consider the limit $n \to \infty$ what does this mean?

$B_n \to \infty$ diverges.

If you keep depositing \$ your account will keep growing.

Recognizing the Geometric Series

– Often in problems you will need to identify a series as being geometric.

Ex $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

we can write this as $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$

so it is geometric w/ $a=1$ and $x=1/2$

The sum $S = \dfrac{a}{1-x} = \dfrac{1}{1-(1/2)} = 2$

this works because $|x|<1$.

From the partial sums: $S_1 = 1$ $\quad S_2 = 1 + 1/2 = 3/2 = 2 - 1/2$

$S_3 = 1 + 1/2 + 1/4 = 7/4 = 2 - 1/4 \quad \dots$

Ex $1 + 2 + 4 + 8 + \dots$

we can write this as $1 + (2) + (2)^2 + (2)^3 + \dots$

What is the sum of this series?

The series has no sum because $|x|>1$ and the partial sums grow without bound.

From the partial sums: $S_1 = 1$ $\quad S_2 = 1+2 = 3$ $\quad S_3 = 1+2+4 = 7$

keep growing!

Ex $6 - 2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \dots$

we can rewrite as $6 - 6\left(\frac{1}{3}\right) + 6\left(\frac{1}{9}\right) - 6\left(\frac{1}{27}\right) + \dots$

Factor out the $a=6$ leading term.

It helps to consider the ratio of successive terms!

$\frac{-2}{6} = -\frac{1}{3}$. $\quad \frac{+2/3}{-2} = -\frac{1}{3}$.

$\frac{-2/9}{2/3} = -1/3$.

or

$6 + 6\left(\frac{-1}{3}\right) + 6\left(\frac{-1}{3}\right)^2 + 6\left(\frac{-1}{3}\right)^3 + \dots$

Bring in the (–) and recognize powers of 3

so $a=6$ and $x = -1/3$

What is the sum? $\quad S = \dfrac{a}{1-x} = \dfrac{6}{1+1/3} = 4.5$

It converges since $|x|<1$.