Calc II Day 21

Last Time: Geometric Series.

Finite $a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}$

Infinite $a + ax + ax^2 + \cdots$

We looked at Partial Sums $S_n = \dfrac{a(1-x^n)}{1-x}$

and saw that the Infinite Series Converges to

$$S = a + ax + ax^2 + \cdots = \frac{a}{1-x} \quad \text{when } |x|<1.$$

Today – we work on generalizing this – a general series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

note for the geom. series $a_n = ax^{n-1}$

Goal – To be able to say when these infinite sums converge or diverge.

General outline:

1. Consider a sequence of Partial Sums $S_1, S_2, S_3 \ldots$
2. Look at what this sequence does as $n \to \infty$
3. If $\lim_{n\to\infty} S_n = S$ then the $\sum_{n=1}^{\infty} a_n$ converges to S.

   if $\lim_{n\to\infty} S_n$ does not exist Then the series diverges.

Ex $\quad a_n = \dfrac{1}{n(n+1)}$ does $\sum_{n=1}^{\infty} a_n$ converge.

First $\quad \sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$

make a conjecture, what do you think? does it converge

These terms seem to be getting smaller so eventually the will go to 0+0+0...

Consider some Partial sums:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

what is the pattern? $\quad S_n = \dfrac{n}{n+1}$

Check this $\quad S_{n+1} = S_n + a_{n+1} = \dfrac{n}{n+1} + \dfrac{1}{(n+1)(n+2)} = \dfrac{n(n+2)+1}{(n+1)(n+2)}$

$$= \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)(n+1)}{(n+1)(n+2)} = \frac{n+1}{n+2} = S_{n+1}$$

by our pattern

Then take the limit.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

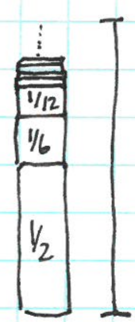
so the series converges to 1

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$$

How does this look visually...

$a_1 + a_2 + a_3 + \cdots$

The sum is like stacking these



as $n \to \infty$ it fills up the area up to 1.

Theorem 9.2 Convergence Properties of series.

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and k is a constant then.

$$\sum_{n=1}^{\infty}(a_n + b_n) \Rightarrow \text{converges to } \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} k a_n \text{ converges to } k \sum_{n=1}^{\infty} a_n$$

2. Changing a finite # of terms in a series does not change whether or not it converges, although it may change the number it converges to.

3. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist then $\sum_{n=1}^{\infty} a_n$ diverges.

4. if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ diverges if $k \neq 0$

Ex Does $\sum_{n=1}^{\infty}(1 - e^{-n})$ converge?

Consider the terms in the series $(1 - e^{-n})$

$$\lim_{n\to\infty}(1 - e^{-n}) = (1 - 0) = 1$$

#3 Garantees divergence but not convergence.

so the series diverges... way out at large n we are adding $1+1+1$ so the sum always grows

Ex $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

the terms $a_n = \frac{1}{n}$ are going to zero as $n \to \infty$

BUT

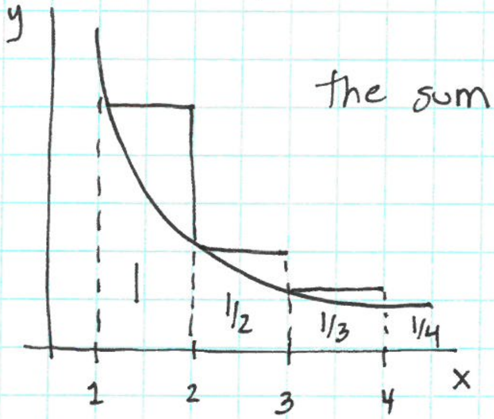
the partial sums look like this...

$S_1 = 1 \quad S_{10} \cong 2.93 \quad S_{100} \cong 5.19 \quad S_{100} \cong 7.49 \quad \ldots$

the sums grow slowly but they keep growing.

Another way to prove convergence or divergence is by comparing sums and improper integrals.

Consider $\int_1^\infty \frac{1}{x}\,dx$



the sum $1 + \frac{1}{2} + \frac{1}{3} + \dots$ is just like LEFT HAND RULE.

Is this over or under the true value of the integral.

$$\int_1^\infty \frac{1}{x}\,dx < \sum_{n=1}^\infty \frac{1}{n}$$

So if $\int_1^\infty \frac{1}{x}$ diverges then the sum is bigger and must also diverge.

Solve the integral:

consider $\int_1^b \frac{1}{x}\,dx = \ln(x)\Big|_1^b = \ln(b) - \ln(1) = \ln(b)$

$\lim_{b\to\infty} \ln(b)$ diverges so $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

Ex Does the $\sum_{n=1}^\infty \frac{1}{n^2}$ converge or diverge.

$\sum_{n=1}^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$

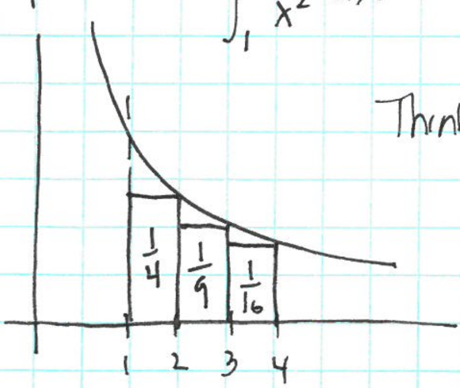
Use a spreadsheet to calculate partial sums...

$S_1 = 1$ $S_{10} = 1.54977$ $S_{100} = 1.6349$ $S_{1000} = 1.64393$

these #'s are leveling out...

I guess converge!

Compare to $\int_1^\infty \frac{1}{x^2}\,dx$



Think about the RIGHT HAND RULE

$$\int_1^\infty \frac{1}{x^2}\,dx > \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$$

Our series is $1 + \underline{\underline{\left[\frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right]}} \cdots$

If the integral converges then the terms in my series converge!

consider

$$\int_1^b \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_1^b = \frac{-1}{b} + 1 \qquad \lim_{b\to\infty} \frac{-1}{b} + 1 = 0 + 1 = 1$$

so $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$$

This means $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges with an upper bound of 2.

Not converges to 2 !!

THE INTEGRAL TEST.

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive

if $\int_1^{\infty} f(x)\,dx$ converges then $\sum a_n$ converges

if $\int_1^{\infty} f(x)\,dx$ diverges then $\sum a_n$ diverges.

<u>Ex</u> THE p-SERIES.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ for what values of p does this series converge?

You tell me!

converges if $p > 1$
diverges if $p \leq 1$

SQUEEZE THEOREM FOR LIMITS.

Let a_n, b_n and c_n be sequences such that for some number M

$$a_n \le b_n \le c_n \text{ for } n > M \text{ and } \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

Then $\lim_{n\to\infty} a_n = L$.

__EX__ Consider the sequence:

$$a_n = \frac{5^n}{n!}$$

hard to know if this converges. Both 5^n and $n!$ grow.

First we think about the sequence...

when n is large $n=100$

which is bigger 5^{100} or $100!$?

Denominator bigger...

Guess $\lim_{n\to\infty} a_n = 0$

NOW SQUEEZE

$$0 \le \frac{5^n}{n!} \quad \text{and} \quad \lim_{n\to\infty} 0 = 0 \checkmark$$

on the other side... $$\frac{5^n}{n!} = \underbrace{\left[\frac{5}{1}\cdot\frac{5}{2}\cdot\frac{5}{3}\cdot\frac{5}{4}\cdot\frac{5}{5}\right]}_{\text{all factors greater than 1}}\underbrace{\left[\frac{5}{6}\cdot\frac{5}{7}\cdots\frac{5}{n}\right]}_{\text{all factors less than 1}}$$

$$\le C\frac{5}{n} \quad \text{where } C = \frac{5}{1}\cdot\frac{5}{2}\cdot\frac{5}{3}\cdot\frac{5}{4}\cdot\frac{5}{5}$$

this is bigger because we removed all the fractions: $\frac{5}{6}\cdot\frac{5}{7}\cdot$

$$0 \le \frac{5^n}{n!} \le C\frac{5}{n} \qquad \lim_{n\to\infty} C\frac{5}{n} = 0$$

so by the squeeze theorem $\lim_{n\to\infty} \frac{5^n}{n!} = 0$