DAY 23

We have been working on Infinite Sums or Series. Today we shift to convergence tests.

Given a series we want to know if it converges ... not necessarily what it converges to.

Positive Series: series such that $a_n > 0$ for all n.

This means that the partial sums form an increasing sequence:

$S_N < S_{N+1}$ Why?

THE INTEGRAL TEST.

Let $a_n = f(n)$, where $f(x)$ is positive, decreasing, and continuous for $x \geq 1$.

(1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

(2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

- It is often easier to just take the integral.

Ex. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ... "The Harmonic Series"

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|b| - \cancel{\ln|1|}^{\,0} = \infty$$

the natural log does not converge.

ASSUMPTIONS MATTER!

$f(x) =$ positive
$f(x) =$ decreasing
$f(x) =$ continuous

$\}$ must have all three!

Ex. Use the Integral Test to answer:

For what values of p does the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

$f(x) = \frac{1}{x^p}$ must be decreasing $p > 0$

then $\int_1^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_1^b \frac{1}{x^p}\, dx =$

$$= \lim_{b \to \infty} \frac{-1}{p+1} x^{-p+1} \Big|_1^b$$

cases $0 < p < 1$ then $\lim_{b \to \infty} \frac{-1}{p+1} b^{-p+1} = \infty$

$-p+1 > 0$

because b^{-p+1} grows.

special case.
$p = 1$ then $\lim_{b \to \infty} \ln|b| = \infty$

$p > 1$ then $\lim_{b \to \infty} \frac{-1}{p+1} b^{-p+1} \rightarrow$ converges

$-p+1 < 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges when } p > 1$$

The Comparison Test.

Assume there exists $M > 0$ such that $0 \le a_n \le b_n$ for $n \ge M$

(i) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

How can we visualize this.

BE CAREFUL... draw a picture [a_n converges but b_n diverges.]

[b_n diverges but a_n converges]

Ex. Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ converge?

the integral test will be hard because $\int \frac{1}{\sqrt{x}\,3^x}\,dx$ is not solved in closed form.

BUT $\frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$

lets investigate $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

Geometric Series: $\sum_{n=1}^{\infty} c r^n$.

TRICK ... we can see that the terms are getting smaller.

$$S_N = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

$$3S_N = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

$S_N - 3S_N = 1$

or $S_N = \frac{1}{2}$

$\lim_{N\to\infty} S_N = \frac{1}{2}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ converges.

Limit Comparison Test. Let a_n and b_n be positive sequences. Assume that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

A • If $L > 0$, then Σa_n converges if and only if Σb_n converges

B • If $L = \infty$ and Σa_n converges then Σb_n converges

C • If $L = 0$ and Σb_n converges then Σa_n converges.

What does all of this mean?

A. $L > 0$ but not $L = \infty$ $\longrightarrow$ $L \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$

L is a finite # meaning these limits are constant multiples... either they both exist or both diverge

B. $L = \infty$ $\longrightarrow$ if $\lim_{n\to\infty} a_n$ converges

then $\lim_{n\to\infty} b_n$ must go to zero

OR $a_n \gg b_n$ ... comparison test.

C. $L = 0$ $\longrightarrow$ if $\lim_{n\to\infty} b_n$ converges then

$\lim_{n\to\infty} a_n$ must go to zero

OR $b_n \gg a_n$ ... comparison test.

Ex Show that $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges.

Choose $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = \frac{1}{n^2}$

so that $\frac{a_n}{b_n} = \frac{n^2}{n^4-n-1} \cdot \frac{n^2}{1} = \frac{n^4}{n^4-n-1}$

$= \frac{n^4}{(n^4)(1-n^{-3}-n^{-4})} = \frac{1}{1-n^{-3}-n^{-4}}$

"I want to show that a_n converges... I choose b_n so that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is easy"

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1-n^{-3}-n^{-4}} = 1$$

so $\sum a_n$ converges if and only if $\sum b_n$ converges...

well $\sum b_n = \sum \frac{1}{n^2}$ this converges !!

So $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges.

1. choose a_n and b_n
2. find the limit $\frac{a_n}{b_n}$
3. decide which part of the theorem applies
4. show that the other half of theorem is satisfied.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$ guess?

as $n \to \infty$ this behaves like $1/n$ and $1/n$ diverges so guess diverge.

let $a_n = \dfrac{n^2-5}{n^3+n+2}$ and $b_n = \dfrac{1}{n}$

then $\dfrac{a_n}{b_n} = \dfrac{1}{1/n} \cdot \dfrac{n^2-5}{n^3+n+2} = \dfrac{n^3-5n}{n^3+n+2}$

$\displaystyle\lim_{n\to\infty} \frac{n^3-5n}{n^3+n+2} = 1$ so $L =$ finite and from the test $\sum a_n$ converges iff $\sum b_n$ converges.

$\sum b_n = \sum 1/n \to$ diverges so $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$ diverges.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ use the comparison test $\dfrac{1}{3^n+1} < \dfrac{1}{3^n} \longrightarrow$ converges.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+3n^3+7}$ use limit comparison.

as $n \to \infty$ a_n acts like $1/n^4$

let $b_n = 1/n^4$

$\dfrac{a_n}{b_n} = \dfrac{n^4}{n^4+3n^3+7} \Rightarrow \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

since $\sum b_n$ converges then $\sum a_n$ converges.