What tools do we have so far?

<u>Rules for Series.</u>

$\sum a_n + b_n$ converges if $\sum a_n$ AND $\sum b_n$ converge

$\sum k a_n$ converges if $\sum a_n$ converges.

Changing a finite # of terms doesn't change convergence

Divergence Test. $\sum a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$

$\sum k a_n$ diverges if $\sum a_n$ diverges.

<u>Integral Test</u> – $a_n = f(n)$ where $f(x)$ is positive decreasing and continuous.

$\int_1^\infty f(x)dx$ converges then $\sum a_n$ converges

$\int_1^\infty f(x)dx$ diverges then $\sum a_n$ diverges.

<u>Comparison Test</u> $0 \le a_n \le b_n$ beyond a certain value of n then

$\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges (bn is the ceiling)

$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges (an is the floor)

<u>Limit Comparison Test</u> compare w/ behavior of a_n as $n \to \infty$

choose b_n based on what a_n "does" at $n \to \infty$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ $c > 0$ and finite

Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

<u>Important Sequences</u>

$\sum_{n=1}^\infty a x^{n-1}$ geom. series converges $|x|<1$

$\sum_{n=1}^\infty \frac{1}{n^p}$ p-series converges $p>1$

- How do we deal with non-positive series?

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges ... very slowly.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots$ converges because the $a_n \to 0$ quickly enough.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges!

Why does this converge? Because the terms are canceling out. The $+-+-$ helps this converge.

<u>ABSOLUTE CONVERGENCE</u> The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Absolute convergence implies convergence... "absolute" is a much stronger condition.

<u>CONDITIONAL CONVERGENCE</u> an infinite series converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

If a series is not absolutely convergent... how can we know if the series converges.

<u>LEIBNIZ TEST for ALTERNATING SERIES</u>

Alternating Series Test.

assume a_n is a positive sequence that is decreasing and converges to zero

$a_1 > a_2 > a_3 > \ldots > 0 \qquad \lim_{n\to\infty} a_n = 0$

Then the alternating series

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{converges.}$$

AND

$0 < S < a_1$

First... one more test for convergence that is widely used.

<u>RATIO TEST.</u> Assume the following limit exists

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

1. If $\rho < 1$ then $\sum a_n$ converges absolutely
2. If $\rho > 1$ then $\sum a_n$ diverges
3. If $\rho = 1$ the test is inconclusive.

<u>Ex</u> Prove that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

1. Compute the ratio: $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$ $\quad a_n = \frac{2^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2}{n+1}$$ here $\frac{2^{n+1}}{2^n} = 2$ $\quad \frac{n!}{(n+1)!} = \frac{1}{n+1}$

2. Take the limit. (of the absolute value)

$$\rho = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$$

so by the ratio test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

Ex. Does $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?

1. Find the ratio $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{(n)^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$

$$= \frac{n^2+2n+1}{n^2} \cdot \frac{1}{2} = \frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

2. Take the limit. (of the absolute value)

$$\rho = \lim_{n\to\infty} \left| \frac{1}{2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\right| = \frac{1}{2}$$

so by the ratio test $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

The hardest part is often the algebra. Work carefully doing algebra before taking the limit.

Ex $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + \frac{1}{2} + \frac{1}{6} + \cdots$

guess – converge since $a_n \to 0$ as $n \to \infty$ faster than $\frac{1}{n}$.

Ratio Test.

$a_n = \frac{1}{n!}$ and $a_{n+1} = \frac{1}{(n+1)!}$ $\quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{1/1\cdot 2\cdot 3\cdot 4\cdot \;\; \cdot n+1}{1/1\cdot 2\cdot 3\cdot 4\cdot \;\; \cdot n}$

$= \frac{1/n+1}{1} = \frac{1}{n+1}$

then $\lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0$ since $\rho < 1$

$\sum_{n=1}^{\infty} \frac{1}{n!}$ converges absolutely.

More examples in class!!