EXAMPLES DAY 24

EX $\sum_{n=1}^{\infty} ne^{-n^2}$ does this converge or diverge.?

1. Investigate the behavior

$$S_1 = 1e^{-1} \sim .36788$$
$$S_2 = e^{-1} + 2e^{-4} \sim .40451$$
$$S_3 = e^{-1} + 2e^{-4} + 3e^{-9} + \cdots \sim .40488$$

2. MY CONJECTURE ... This series converges.

3. $a_n = ne^{-n^2}$ this is a continuous function that I think I can integrate ... Integral test.

GOAL ... show that $\int_1^{\infty} f(x)\,dx$ converges

$f(x) = xe^{-x^2}$ $\quad \underbrace{\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^{b} xe^{-x^2}\,dx}_{\text{Improper}}$.

u-sub.

evaluate the integral. $\int_1^{b} xe^{-x^2}\,dx$

let $u = x^2$ $\quad x=1 \rightarrow u=1$
$du = 2x\,dx$ $\quad x=b \rightarrow u=b^2$
$dx = \frac{du}{2x}$

$\int_1^{b^2} \frac{1}{2}e^{-u}\,du \rightarrow$

$$= -\frac{1}{2}e^{-u}\Big|_1^{b^2} = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}$$

take the limit. $\lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2} + \frac{1}{2e}\right] = \frac{1}{2e}$ converges.

FINAL STATEMENT $\rightarrow$ Since $\int_1^{\infty} f(x)\,dx$ converges then by the integral test $\sum_{n=1}^{\infty} a_n$ converges.

OR By the integral test $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

EX $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n-1}}$ does this converge or diverge?

1. Investigate ...

$$S_1 = \frac{1}{\sqrt{2}} \sim .70711$$

$$S_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{11}} \sim 1.00862$$

$$S_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{32}} \sim 1.18539$$

2. MY CONJECTURE ... this series converges.

3. $a_n = \dfrac{1}{\sqrt{n^3+2n-1}}$ this is continuous BUT would be a nightmare to integrate!

can I think of a series that is larger but also converges ... NOTE ... the importance of my conjecture!

$$\frac{1}{\sqrt{n^3+2n-1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

"take away some of the denominator ... or make the denominator smaller"

BUT $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is the p-series which we proved converges for $p > 1$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

FINAL STATEMENT. thus by the comparison test since $\dfrac{1}{\sqrt{n^3+2n-1}} \leq \dfrac{1}{\sqrt{n^3}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n-1}}$ also converges.

Ex $\sum_{n=2}^{\infty} \frac{n^3}{\sqrt{n^7+2n^2+1}}$ does this converge or diverge?

NOTE we can use l'Hopitals rule to show that $a_n \to 0$ this ONLY means that the series Might Converge (could possibly converge).

1. Investigate ...
this one is harder to write out ... I used a spreadsheet program (Excel) to calculate to S_{24}.
As n gets bigger the biggest terms will dominate

$$a_n = \frac{n^3}{\sqrt{n^7+2n^2+1}} = \frac{n^3}{n^3\sqrt{n+2n^{-4}+n^{-6}}} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}}$$

here $n^7+2n^2+1 = n^6(n+2n^{-4}+n^{-6})$
I factor out n^6 and $\sqrt{n^6} = n^3$

2. Conjecture: I think this will Diverge.

3. $a_n = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}}$
- continuous but not easy to integrate...
- For comparison test we would need something smaller that diverges .... hard to find.

Use the limit comparison test ... we want case 1 $L>0$.

Choose $b_n = \frac{1}{\sqrt{n}}$ because I know this diverges AND $\frac{a_n}{b_n}$ will simplify so that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is easy.
behavior in limit.

simplify $\frac{a_n}{b_n} = \frac{1}{\sqrt{n+2n^{-4}+n^{-6}}} \cdot \frac{\sqrt{n}}{1} = \frac{\sqrt{n}}{\sqrt{n}\sqrt{1+2n^{-5}+n^{-7}}}$

$$n+2n^{-4}+n^{-6} = n(1+2n^{-5}+n^{-7})$$

so $\frac{a_n}{b_n} = \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}}$ my goal here was to make it so in the limit as $n \to \infty$ the denominator is easy to deal with!

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+2n^{-5}+n^{-7}}} = 1$$

By the limit comparison test, if $L > 0$ then

$\sum a_n$ converges if and only if $\sum b_n$ converges

But we know $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges

therefore $\sum a_n = \sum \frac{n^3}{\sqrt{n^7+2n^2+1}}$ diverges.

It is okay to make the wrong conjecture ... you will find yourself unable to prove it ... the theorems wont work ... you will go in circles

If this happens, try changing your conjecture and prove that.

Ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{10n+1}$

This is an alternating series so need $a_n \to 0$ as $n \to \infty$ for convergence.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{10n+1} = \frac{1}{10} \neq 0$ since this is not zero the series diverges.

FINAL STATEMENT – The series diverges.

Ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^4}{e^n}$

Alternating: $\lim_{n\to\infty} \frac{n^4}{e^n} = 0$ so this converges. Lets do the ratio test to see if it converges absolutely

$a_n = \frac{n^4}{e^n}$ $\quad a_{n+1} = \frac{(n+1)^4}{e^{n+1}}$ $\quad \frac{a_{n+1}}{a_n} = \frac{(n+1)^4}{n^4} \cdot \frac{e^n}{e^{n+1}} = \frac{1}{e} \frac{(n+1)^4}{n^4}$

$\lim_{n\to\infty} \left| \frac{1}{e} \frac{(n+1)^4}{n^4} \right| = \left| \frac{1}{e} \right| = \frac{1}{e} < 1$

since this is less than one the series converges absolutely.

FINAL STATEMENT. By the ratio test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^4}{e^n}$ converges absolutely.