10.6 POWER SERIES.　　　Day 25

A power series is an infinite series of the form:

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

c is called the center and x is a variable.

The power series may converge for some x and diverge for others.

When do the series converge... lets look at an example.

Power Series. Day 25.

$$F(x) = \sum_{n=0}^{\infty} b_n (x-c)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$$

what is the center ? $c = 0$
what is general constant? $b_n = \frac{1}{2^n}$

notice that when $x = 0$ $F(0) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{2^n}$

converges

<u>but</u>

when $x = 2$ $F(2) = 1 + 1 + 1 + 1 \dots$ diverges.

Let's try a ratio test on this.

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \qquad a_{n+1} = \frac{x^{n+1}}{2^{n+1}} \qquad a_n = \frac{x^n}{2^n}$$

include the x's in the ratio test!

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} = \frac{x}{2}$$

$$\lim_{n\to\infty} \left| \frac{x}{2} \right| = \lim_{n\to\infty} \frac{1}{2}|x|$$

by the ratio test this

Converges when $\frac{1}{2}|x| < 1 \Rightarrow |x| < 2$ $\quad -2 < x < 2$

Inconclusive when $\frac{1}{2}|x| = 1 \Rightarrow |x| = 2$ $\quad x = \pm 2$

Check the inconclusive points directly

$F(2) = 1 + 1 + 1 + 1 + \dots$ Diverges
$F(-2) = 1 - 1 + 1 - 1 + 1 \dots$
Diverges because it is either 0 or 1 depending on how many terms we have.

So we say the Power Series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \quad \text{converges when} \quad |x| < 2$$

The number 2 here is a very important number !!!

We call it the <u>RADIUS OF CONVERGENCE</u>.

The Radius of Convergence: Every power series.

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$ has a radius of convergence

denoted R, which is either a non-negative number, $R \geq 0$, or infinity, $R = \infty$.

1. $R \geq 0$ then $F(x)$ converges absolutely when $|x-c| < R$ and diverges when $|x-c| > R$

~~xxxxx~~ $R = 0 \Rightarrow$ convergence for $x = c$ only.

notice we cant say what happens at $|x-c|=R$

2. $R = \infty$ the $F(x)$ converges absolutely for all x.

• When R fails to exist this tells us nothing.

Ex. Where does $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n} (x-5)^n$ converge?

What is the center? $C = 5$

What is the general constant? $b_n = \frac{(-1)^n}{4^n n}$

To find the radius of convergence we use the ratio test..

$$a_{n+1} = \frac{(-1)^{n+1}}{4^{n+1}(n+1)} (x-5)^{n+1} \qquad a_n = \frac{(-1)^n}{4^n (n)} (x-5)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(x-5)^{n+1}}{(x-5)^n} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{n}{n+1} = -1(x-5) \cdot \frac{1}{4} \cdot \frac{n}{n+1}$$

expect 4 terms

$$\lim_{n\to\infty} \left| -\frac{1}{4}(x-5)\frac{n}{n+1} \right| = \lim_{n\to\infty} \frac{1}{4}|x-5|\frac{n}{n+1} = \frac{1}{4}|x-5|$$

By the ratio test this converges when

$$\rho = \frac{1}{4}|x-5| < 1 \quad \text{or} \quad |x-5| < 4$$

$$\text{or} \quad -4 < x-5 < 4$$

$$\text{or} \quad 1 < x < 9$$

The radius of convergence is $R=4$
and $F(x)$ converges absolutely on the open intervale $(1, 9)$

Now check the endpoints ... $\rho = 1$ the test is inconclusive.

when $x=1$

Plug in the endpoints and do the convergence tests!

$$F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n \cdot n}(-4)^n = \sum_{n=1}^{\infty} \frac{4^n}{4^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

this is the harmonic series and we know it diverges!

when $x=9$

$$F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

here $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series.

Recall our friend Liebniz ... if $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

then the alternating series converges!

$F(9)$ converges by the Liebniz Test.

Ex. Where does the Power Series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$ converge? YOU TRY ...

$$a_{n+1} = \frac{x^{2(n+1)}}{(2(n+1))!} \qquad a_n = \frac{x^{2n}}{(2n)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)!} = x^2 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$$

$$= \frac{x^2}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$$

So $\rho = 0$ for all x and the radius of convergence:

$$R = \infty$$

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MATH 122 - Activity Day 25

Exploring Power Series.

A. Consider the series:

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots$$

1. What is the center for this series?
2. Write down the general term for this series.
3. On a computer go to desmos.com/calculator and graph the first seven partial sums on the same graph:

$$S_1 = (x-1), \quad S_2 = (x-1) - \frac{(x-1)^2}{2}, \quad S_3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \dots$$

4. Discuss what you see. For what x-values do the partial sums seem to be converging (approaching the same number)? For what x-values are the partial sums diverging?
5. Now use the Ratio Test to find the Radius of Convergence.
6. How does the Radius of Convergence compare to your graph?

Now repeat this exercise for the following examples:

B.

$$\frac{(x-1)}{3} - \frac{(x-1)^2}{2 \cdot 3^2} + \frac{(x-1)^3}{3 \cdot 3^3} - \frac{(x-1)^4}{4 \cdot 3^4} + \dots$$

C.

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

NOTE: You may need to graph more than S_7 to get a better idea of what is happening for $x < 0$.

D.

$$1 + 2^2x^2 + 2^4x^4 + 2^6x^6 + \dots$$

NOTE: You will need to be careful how you write your general term.
In order to use the ratio test you need $n = 1, 2, 3, 4, \dots$ not skipping values.