Before we start...

A few more examples: Power Series $F(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$

Consider: $F(x) = 1 + 2^2x^2 + 2^4x^4 + 2^6x^6$

what is the generic term? $2^n x^n \quad n = 0, 2, 4, 6 \ldots$

It is tempting to write this!
Problem: We can't use the RATIO test without a_n and a_{n+1}.

Instead we should write $2^{2n} x^{2n} \quad n = 0, 1, 2, \ldots$

$$F(x) = \sum_{n=0}^{\infty} 2^{2n} x^{2n}$$ now use the ratio test

$a_n = 2^{2n} x^{2n}$ and $a_{n+1} = 2^{2n+2} x^{2n+2}$

then $$\frac{a_{n+1}}{a_n} = \frac{2^{2n+2} x^{2n+2}}{2^{2n} x^{2n}} = 2^2 x^2 = 4x^2$$

need $\lim_{n\to\infty} |4x^2| < 1$ or $|x^2| < \frac{1}{4}$ or $-\frac{1}{2} < x < \frac{1}{2}$

Radius of convergence $R = 1/2$
with a center $c = 0$

You Try: $F(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

Find the general term: $\frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \quad n = 1, 2, 3 \ldots$

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n+2-1}}{x^{2n-1}} \cdot \frac{(2n-1)!}{(2n+2-1)!} = x^2 \cdot \frac{1}{(2n+1)\cdot(2n)} \qquad \lim_{n\to\infty} \left| \frac{x^2}{(2n+1)(2n)} \right| = 0$$

Radius of convergence $R = \infty$

Day 26

Last time... Power Series and the Ratio Test.

Recall Geometric Series:

$$\sum_{n=0}^{\infty} cx^n = \frac{c}{1-x} \quad \text{converges when } |x|<1$$

From Before... How do we show this... look at the partial sums

$$\begin{cases} S_N = c + cr + cr^2 + cr^3 + \cdots + cr^N \\ rS_N = cr + cr^2 + cr^3 + \cdots + cr^{N+1} \end{cases}$$

consider a strategic constant multiple

subtract these...

$$S_N - rS_N = c + cr - cr + cr^2 - cr^2 \cdots - cr^{N+1}$$

$$S_N(1-r) = c - cr^{N+1}$$

$$S_N = \frac{c - cr^{N+1}}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{c - cr^{N+1}}{1-r} = \frac{c}{1-r}$$

as long as $|r|<1$

So given this how might we write: $f(x) = \frac{1}{1-x}$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$$

$r = x$
$c = 1$

and it converges when $|x|<1$.

This helps us write lots of functions as power series.

Ex Write $f(x) = \frac{1}{1-2x}$ as a power series.

- when the function is close to $\frac{1}{1-x}$ we just rewrite it.

$$\frac{1}{1-2x} = \underset{\text{let } u = 2x}{\frac{1}{1-u}} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

converges when $|2x| < 1$
or $|x| < \frac{1}{2}$ ✓

YOU TRY...

Ex Write $\frac{1}{1+x^2}$ as a power series.

$$\frac{1}{1+x^2} = \underset{\text{let } u = -x^2}{\frac{1}{1-(-x^2)}} = \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

converges when $|x| < 1$

Theorem Term-by-Term Differentiation and Integration:

assume that $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

has a radius of convergence $R > 0$. Then $F(x)$ is differentiable on $(c-R, c+R)$. We may integrate and differentiate term by term:

$$F'(x) = \sum_{n=0}^{\infty} a_n\, n(x-c)^{n-1}$$ took the derivative

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$ integrated

A: constant of integration.

A TRICK!

<u>Ex</u> find the power series for $f(x) = \dfrac{1}{(1-x)^2}$ using differentiation.

Here we know that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{so} \quad \frac{1}{(1-x^2)} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x^2)} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

REMEMBER OUR GOAL ... we want to be able to approximate numbers and functions with series!

FUN APPLICATION.

I CLAIM: $\pi = 4 - \dfrac{4}{3} + \dfrac{4}{5} - \dfrac{4}{7} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$

We can show this!

We just wrote $f(x) = \dfrac{1}{1+x^2}$ as a power series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

<u>AND</u>

we found out that we can integrate term-by-term.

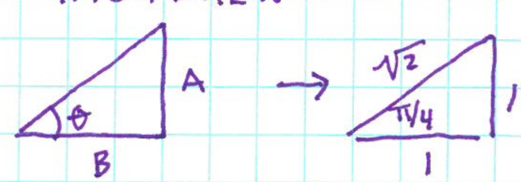
So what is ... $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + c$

this is arctan!

$$\frac{\pi}{4} = \arctan(1) = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$\arctan(0) = 0$$

TRIG REVIEW



$\tan\theta = \dfrac{A}{B}$ $\qquad \theta = \arctan\left(\dfrac{A}{B}\right)$ $\qquad$ 45-45-90

We can use this!

$$\frac{\pi}{4} = \arctan(1) = \int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

technically this is improper

we know this converges for $|x|<1$ but when $x=1$? test our series...

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by the LIEBNIZ TEST!

so

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

WE DID IT!!

Big Example:

Ex Write as a powerseries and prove interval of convergence.

$$f(x) = \frac{1}{16+2x^3}$$

FIRST WRITE AS POWER SERIES.

1. rewrite as $\frac{1}{1-x}$ ...

$$\frac{1}{16+2x^3} = \frac{1}{16}\cdot\frac{1}{\left(1+\frac{x^3}{8}\right)} = \frac{1}{16}\cdot\frac{1}{\left(1-\left(\frac{-x^3}{8}\right)\right)} = \frac{1}{16}\cdot\frac{1}{1-u}$$

first work on the 1

then get the minus sign

let $u = \frac{-x^3}{8}$

this gives u.

2. put in sum form...

$$= \frac{1}{16}\sum_{n=0}^{\infty} u^n = \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n = \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^n}$$

write as u- power series

sub back in $u = \frac{-x^3}{8}$

simplify to standard form.

3. Answer $$\boxed{\frac{1}{16+2x^3} = \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^n}}$$

NOW CONSIDER CONVERGENCE

Ratio Test $\quad a_{n+1} = \frac{(-1)^{n+1}x^{3(n+1)}}{8^{n+1}} \qquad a_n = \frac{(-1)^n x^{3n}}{8^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(-1)^n}\,\frac{\cancel{x}^{3n+3}}{x^{3n}}\cdot\frac{8^n}{8^{n+1}} = (-1)\cdot x^3\cdot\frac{1}{8}$$

$$\rho = \lim_{n\to\infty}\left|\frac{-x^3}{8}\right| = \frac{|x^3|}{8} < 1 \qquad |x^3| < 8$$

because $2^3 = 8$

or $|x| < 2$

so RADIUS OF CONVERGENCE $R=2$

Interval: $-2 < x < 2$

still need to check end points. $x=2 \quad x=-2$

$$F(2) = \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{3n}}{8^n} = \frac{1}{16}\left(\sum_{n=0}^{\infty}\frac{(-1)^n (2^3)^n}{8^n}\right) = \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n$$ does not converge.

LIEBNIZ

$$F(-2) = \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n\frac{(-2)^{3n}}{8^n} = \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{3n}(2^3)^n}{8^n} = \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n(-1)^{3n} = \frac{-1}{16}\sum_{n=0}^{\infty} 1$$

does not converge.

FINAL ANSWER.

writen as a power series $\dfrac{1}{16+2x^3} = \dfrac{1}{16}\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n}}{8^n}$

which has an interval of convergence $-2<x<2$.