10.1 Taylor Polynomials.

Last Time ... we can express functions as a power series...

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{converges } |x|<1$$

we did this by comparing with the sum of the geom. series.

This gives us a limited set of functions that we can write as power series...

$$\frac{7}{3+2x^2} = \frac{7}{3(1+\frac{2}{3}x^2)} = \frac{7}{3}\frac{1}{(1-(-\frac{2x^2}{3}))} = \frac{7}{3}\sum_{n=0}^{\infty}\left(-\frac{2x^2}{3}\right)^n = \frac{7}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n x^{2n}$$

converges when $\left|\frac{-2x^2}{3}\right| < 1$

$$|x^2| < \frac{3}{2}$$

$$|x| < \sqrt{\frac{3}{2}}$$

to prove this ... RATIO TEST — RADIUS OF CONVERGENCE

$$a_n = \frac{2^n x^{2n}}{3^n} \qquad a_{n+1} = \frac{2^{n+1}x^{2n+2}}{3^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}x^{2n+2}}{3^{n+1}}}{\frac{2^n x^{2n}}{3^n}} = \frac{2^{n+1}}{2^n}\cdot\frac{x^{2n+2}}{\cancel{x}^{2n}}\cdot\frac{3^n}{3^{n+1}} = 2\cdot x^2\cdot\frac{1}{3} = \frac{2}{3}x^2$$

$$\lim_{n\to\infty}\left|\frac{2}{3}x^2\right| = \left|\frac{2}{3}x^2\right| < 1 \quad \text{or} \quad x^2 < \frac{3}{2} \qquad -\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$$

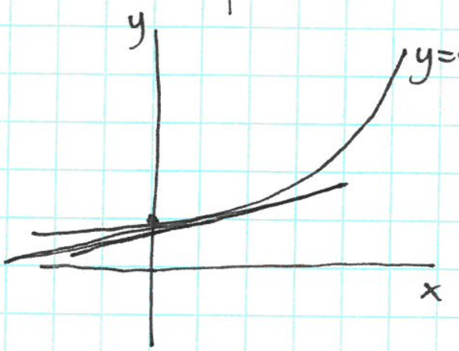
same answer!

NOW... what if we wanted to express other functions as a power series e^x, $\sin(x)$, $\cos(x)$, etc.

can't rewrite these as $\frac{1}{1-x}$!

EX
Consider $y=e^x$ lets try to write out a few polynomial approximations to this.

Approximate $y=e^x$ with a straight line. that matches at $x=0$.



when $x=0$ $y=e^0=1$

straight line: $y=mx+b$

m ↓ slope = derivative of y $= y' = e^x$

$m=e^0=1$

$y=x+b$ through the point $(0,1)$

GRAPH ON DESMOS...

$1=b$ so $y=1+x$ is a linear approximation to e^x.

Does a good job near $x=0$!!

Problem... doesn't curve... how could we do better?

Parabola that goes throug $(0,1)$ matches the slope and the curvature!

slope $= f'(0)$
curvature $= f''(0)$

Turns out

$y=1+x+\frac{x^2}{2}$ does this...

↑ what does this look like? POWER SERIES!

Lets see if we can generalize this!

$$\text{General Power Series} = F(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$$

given a function $f(x)$ we want

$$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

"Power Series centered at c"

hardest part: finding $b_0, b_1, \ldots$

Lets take some derivatives of this.

$$f'(x) = b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots$$

$$f''(x) = 2b_2 + 3\cdot 2\cdot b_3(x-c) + \cdots$$

$$f'''(x) = 3\cdot 2\cdot 1\cdot b_3 + \cdots$$

wait! I can plug in $x=c$ and solve for $b_1, b_2, b_3 \ldots$

$f'(c) = b_1 + 0 + 0 + \cdots$ so $b_1 = f'(c)$

$f''(c) = 2\cdot 1\cdot b_2 + 0 + \cdots$ so $b_2 = \dfrac{f''(c)}{2\cdot 1}$

$f'''(c) = 3\cdot 2\cdot 1\cdot b_3$ so $b_3 = \dfrac{f'''(c)}{3\cdot 2\cdot 1}$

guess: $b_4 = \dfrac{f''''(c)}{4\cdot 3\cdot 2\cdot 1}$ $\qquad b_n = \dfrac{f^{(n)}(c)}{n!}$

We can write ANY function $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

as long as I can take derivatives $f^{(n)}(c)$.

WOW!!!

The ~~[illegible]~~ Partial Sums of this series are called TAYLOR POLYNOMIALS.

Back to our example.

$f(x) = e^x$

write out the first few taylor polynomials. centered at $c=0$

$P_1 = 1+x \quad = \quad f(0) + f'(0)(x-0) = 1 + 1x = 1+x.$

we did this linear.

$$P_2 = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + x * \frac{x^2}{2}$$

$$P_3 = 1 + x + \frac{x^2}{2} + \frac{f'''(0)}{3!}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Look at the Graph on DESMOS!!

When $c=0$ this is called the Maclaurin Series.

Vocab: P_n $\quad$ n^{th} order Taylor Polynomial (assuming it contains an x^n term)

$c = 0$ $\quad$ centered about $x=0$

Ex. Find the fourth order Taylor polynomial centered about $x=1$ for the function $f(x) = e^x$

$$P_4 = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4$$

Graph on DESMOS — what do you see?
what is the difference?

TOMORROW.

NOTE.. we still need to talk about convergence!
we still need to discuss the full

SERIES $\sum_{n=0}^{\infty} \ldots$ $\quad$ right now there are just partial sums.

Worksheet.

For $y = \cos x$ near $x = 0$

$\cos x$	1
$-\sin x$	0
$-\cos x$	-1
$\sin x$	0
$\cos x$	1
$-\sin x$	0
$-\cos x$	-1
$\sin x$	0

Table of values.

$$P_1 = 1 + 0$$

$$P_2 = 1 + 0 - \frac{1(x)^2}{2}$$

$$P_3 = 1 + 0 - \frac{x^2}{2} + 0$$

$$P_4 = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24}$$

$$P_5 = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24} + 0$$

$$P_6 = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24} + 0 - \frac{x^6}{6!}$$

$$P_7 = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24} + 0 - \frac{x^6}{6!} + 0$$

Graph... approximation gets better!

The general term would be $\rightarrow$ $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n!}$

P_∞ should match perfectly...

Ratio test $\quad \frac{a_{n+1}}{a_n} = \frac{x^{2n+2}}{x^{2n}} \cdot \frac{2n!}{2n+2!} = x^2 \cdot \frac{1}{(2n+2)(2n+1)}$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0 < 1$$

converges for all x-values

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n!}$$

Worksheet.

For $y=\ln(x)$ near $x=1$

$\ln(x)$	0
$1/x$	1
$-1/x^2$	-1
$2/x^3$	2
$-6/x^4$	-6
$24/x^5$	24
$-120/x^6$	-120

$$P_6 = (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \frac{120(x-1)^6}{6!}$$

general term: $\dfrac{(-1)^{n+1}(n-1)!\,(x-1)^n}{n!}$

$$= \frac{(x-1)^n}{n}(-1)^{n+1}$$

Graph ... seems to converge for $0 < x < 2$!!

Ratio Test: $\dfrac{a_{n+1}}{a_n} = \dfrac{(x-1)^{n+1}}{(x-1)^n} \cdot \dfrac{n}{n+1} = \dfrac{n}{n+1}(x-1)$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1| < 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2 \quad \checkmark$$