DAY 28

Finish Workshet From Last Class.

Taylor Series — the infinite sum of terms or an infinite Taylor Polynomial.

From Last Time. $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

general term

How do we get this?

1. Remember the formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-a)^n$

   general Taylor Series.

2. Find derivatives of the given function, plug in the center c.

3. Write out the terms in the series using the formula.

4. Look for a pattern and write down the general series for the specific function.

Now lets add the final piece! Convergence.
For what values of x does the series converge?

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ use the ratio test! $a_n = \frac{x^n}{n!}$ $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} \cdot \frac{(n!)}{(n+1)!} = \frac{x}{n+1}$$

$\lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$ and $0 < 1$ so the series converges FOR ALL X-VALUES.

Show m-file.

Examples from the worksheet:

$$f(x) = \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

lets test convergence: $a_n = \frac{x^{2n}}{(2n)!}$ $a_{n+1} = \frac{x^{2n+2}}{(2n+2)!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)!} = \frac{x^2}{(2n+2)(2n+1)}$$

$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0$ and $0<1$ converges for all x!

$f(x) = \ln(x)$ about $x=1$ $= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n}$

lets test convergence: $a_n = \frac{(x-1)^n}{n}$ $a_{n+1} = \frac{(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \cdot \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| \frac{(x-1)n}{n+1} \right|$$

$= |x-1| < 1$

for convergence.

$-1 < x-1 < 1$ or $0 < x < 2$ The series only converges between $0<x<2$.

End points?

$f(0) = \sum_{n=0}^{\infty} (-1)^n (-1)^n \cdot \frac{1}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

$f(2) = \sum_{n=0}^{\infty} (-1)^n (1)^n \cdot \frac{1}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ converges.

$\ln(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n}$ is only a good approximation on $0 < x \leq 2$.

An important function: BINOMIAL SERIES.

Find the taylor series of $f(x) = (1+x)^p$ about the point $x=0$.

Derivatives	About $x=0$
$(1+x)^p$	1
$p(1+x)^{p-1}$	p
$p(p-1)(1+x)^{p-2}$	$p(p-1)$
	$p(p-1)(p-2)$
	$\vdots$ notice a pattern.

so

$$\underbrace{(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots}_{\text{This is the BINOMIAL SERIES.}} \quad \sum_{n=0}^{\infty} x^n \frac{p(p-1)(p-2)\cdots(p-n}{n!}$$

converges. $-1 < x < 1$

It gives the same result as multiplying $(1+x)^p$ out!

Ex
$$(1+x)^5 = 1 + 5x + 10x^2 + \frac{5\cdot4\cdot3}{3\cdot2\cdot1}x^3 + \frac{5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot1}x^4 + \frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}x^5$$
$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Interesting... binomial coef are symmetric.
related to pascals Δ.

Ex Use the Binomial Series to write down the expansion of $f(x) = \sqrt{1+x}$ about $x=0$.

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{x^3}{3!} + \cdots$$
$$= 1 + \frac{1}{2}x + \frac{-1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} + \cdots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$