An important function: BINOMIAL SERIES.

Find the taylor series of $f(x) = (1+x)^p$ about the point $x=0$.

Derivatives	About $x=0$
$(1+x)^p$	1
$p(1+x)^{p-1}$	p
$p(p-1)(1+x)^{p-2}$	$p(p-1)$
	$p(p-1)(p-2)$
	$\vdots$ notice a pattern.

so

$$\underbrace{(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots}_{\text{This is the BINOMIAL SERIES.}} \quad \sum_{n=0}^{\infty} x^n \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

converges $-1 < x < 1$

It gives the same result as multiplying $(1+x)^p$ out!

Ex $$(1+x)^5 = 1 + 5x + 10x^2 + \frac{5\cdot4\cdot3}{3\cdot2\cdot1}x^3 + \frac{5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot1}x^4 + \frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}x^5$$
$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Interesting... binomial coef are symmetric.
related to pascals Δ.

Ex Use the Binomial Series to write down the expansion of $f(x) = \sqrt{1+x}$ about $x=0$.

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \tfrac{1}{2}x + \left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\frac{x^2}{2!} + \left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\frac{x^3}{3!} + \cdots$$
$$= 1 + \tfrac{1}{2}x + \frac{-1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} + \cdots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

So far:

General Formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

c = center "expand $f(x)$ about the point $x=c$"

Ones that we know:

Maclaurin Series $c=0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$$

$-\infty < x < \infty$

About $x=1$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

$0 < x \leq 2$

GEOMETRIC SERIES

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

BINOMIAL SERIES

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{x^n}{n!} p(p-1)(p-2)\cdots(p-n+1)$$

- We can always check the radius of convergence using RATIO TEST.
- Given a new function, what is the hardest part of writing down the Taylor Series?

  Taking Derivatives and plugging in!

Today — Tricks for making this easier!

New Series by Substitution:

<u>Ex</u> Find the taylor series for $f(x)=e^{-x^2}$ expanded about $x=0$.

Think about these derivatives... messy!

But, we know
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
lets use this.

then
$$e^{(-x^2)} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

YOU TRY: Find the taylor series for $f(x) = \sin(x^3)$ expanded about $x=0$.

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{6n-3}}{(2n-1)!}$$

A harder example:
expand $f(x) = e^{\sin\theta}$ about $\theta=0$

we know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$e^{\sin\theta} = 1 + (\sin\theta) + \frac{(\sin\theta)^2}{2!} + \frac{(\sin\theta)^3}{3!} + \cdots$$
now plug in for $\sin\theta$
$$= 1 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + \frac{1}{2}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2 + \frac{1}{3!}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^3 + \cdots$$
then gather like powers of θ
$$= 1 + \theta + \frac{1}{2}\theta^2 + \left[\cancel{-\frac{\theta^3}{3!} + \frac{\theta^3}{3!}}^{\,0}\right] + \cdots$$

* much harder to write down general term.
* often a few terms are enough.

Ex Find the Taylor Series Expansion of

$$f(x) = e^x \cos(x) \quad \text{for } x \text{ near } 0.$$

We can multiply two know series... just be careful.

1. write out the series to the same degree (here I will choose 5)

$$e^x = \left(1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)$$

$$\cos x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$$

note the x^5 term here is multiplied by zero... see p3 notes.

2. Do the multiplication... term by term.

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)$$

distribute each term in:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right) - \frac{x^2}{2}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)$$

$$+ \frac{x^4}{24}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)$$

distribute again... this time only keep terms to 5th order

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right) - \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{4} + \cdots\right)$$

$$+ \left(\frac{x^4}{24} + \frac{x^5}{24} + \cdots\right)$$

simplify: $1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30}$

3. Soln $\quad e^x \cos(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots$

no general form needed.

Lets use these on some more interesting problems!!

<u>Ex</u> Find $\int_0^1 e^{x^2}\,dx$.

We can't do this exactly, but we can use a
Antiderivative Taylor Series.

First: expand e^{x^2} about $x=0$ from limits.

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

Plug in and integrate.

$$\int_0^1 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \, dx$$

term by term.

$$= x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \cdots \Big|_0^1$$

$$= 1 + \frac{1}{3} + \frac{1}{5\cdot 2!} + \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)\cdot n!}$$

<u>Ex</u> Approximate the value of $\sin(.1)$ without using a calculator.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Then $\sin(.1) = .1 - \frac{(.1)^3}{3!} + \frac{(.1)^5}{5!} - \cdots$

To approximate we TRUNCATE the series ... This results in truncation error but gives us a numerical approximation.

$$\sin(.1) \approx .1 - \frac{1}{6}(.001) = .1 - (.166\overline{6})(.001) = .1 - .000166$$
$$= .099834$$

$$\sin(.1) = .0998334 1664$$

correct to here.