Today: Applications of Taylor Series and Error.

Last time: $\int_0^1 e^{x^2}\,dx$.

We can also use this to approximate functions.

<u>Ex</u> Use the Taylor Series to approximate $\sin(.1)$.

1. center the series near the point you want to approximate – but somewhere convienient. Here choose $c=0$ since we know $\sin(0)=0$

2. Expand $\sin(x)$ near $x=0$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

3. Plug in the point.

$$\sin(.1) = .1 - \frac{(.1)^3}{3!} + \frac{(.1)^5}{5!} - \cdots$$

4. Truncate the series for a numerical approximation.

$$\sin(.1) \cong .1 - \frac{(.1)^3}{3!} = 0.09983\bar{3}$$

plug in $\sin(.1)$ on calculator $= 0.0998334166\,4$.
note: your calculator makes an approximation too!! Numerical Analysis.

Here when we chopp off or Truncate the series we introduce error...

Ex Eulers Formula.

In many upper level math/physics classes you see the formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$ lets show that this is true.

first $i=\sqrt{-1}$ so $i^2=-1$ and $i^3 = -\sqrt{-1} = -i$ and $i^4 = 1$

then $e^{i\theta}$ can be expanded as a Taylor series...

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots$$

gather like terms with i

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos(\theta) + i\sin(\theta).$$

The formula works!

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

There are SO MANY APPLICATIONS OF TAYLOR SERIES !!

- Nonlinear Dynamics and Chaos — "Linearize" hard functions. Use the Taylor Series and only keep linear terms.

- Numerical Analysis — This is how we can derive SIMPSONS RULE and ERROR

derivatives $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

Derived from Taylor Series.

- Physics – Special Relativity

$E = mc^2$ is an approximation for low velocities.

really $E = \gamma mc^2$ and $\gamma = \dfrac{1}{\sqrt{1 - v^2/c}}$

Taylor Expand $\gamma(v)$ near $v = 0$

$$\gamma = \underbrace{\left(1 + \left(-v^2/c\right)\right)^{-1}}_{\text{Binomial}} = 1 + \frac{v^2}{c} + \cdots \qquad \gamma \approx 1$$

and $E = mc^2$!

- Digital graphing or photo reconstruction

Many methods use smoothing or repairing functions that are derived using Taylor Series (Polynomials)

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Now we need to consider Error in using the Taylor Polynomials to approximate functions.

LAGRANGE ERROR BOUND...

Assume $f(x)$ and it's derivatives are continuous. If P_n is the n^{th} Taylor polynomial for $f(x)$ about a then

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $|f^{(n+1)}| \leq M$ between a and x.

<u>Ex</u> Give an error bound when $f(x) = e^x$ is approximated by P_4 centered about $x=0$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4} = P_4(x)$$

The Error

$$|E_4(x)| = |f(x) - P_4(x)| \leq \frac{M}{5!}|x-a|^5 = \frac{M}{5!}x^5$$

now find M: $f^{(5)}(x) = \frac{d^5}{dx^5}e^x = e^x$

$$|E_4(x)| \leq \underbrace{\frac{|x|^5 e^x}{5!}}_{\text{when is this the largest?}} \leq \frac{\sqrt{e}|x|^5}{5!} < \frac{2|x|^5}{120} < \frac{2(\frac{1}{2})^5}{120}$$

Bounds the error.

→ when $x = \frac{1}{2}$ e^x is large.

the approximation has an error of at most

$$|E_4(x)| < 0.0006$$

We showed that the Taylor Series converged for all x in the case of e^x, $\sin(x)$, $\cos(x)$.

Now we can show that these series actually converge TO THE CORRECT FUNCTION.

Ex $\cos(x) = 1 - x^2/2! + x^4/4! - \cdots$

$|E_n(x)| = \cos(x) - P_n(x)$ so for equality we need $E_n \to 0$ as $n \to \infty$

use the Lagrange Error Bound:

$$|E_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{above } a=0$$

and $M \geq |f^{(n+1)}(x)|$ this is either $\pm\cos(x)$ or $\pm\sin(x)$

worst case $M = 1$

$$|E_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$ does this go to zero as $n \to \infty$?

consider $\frac{|E_{n+1}|}{|E_n|}$ if this is less than 1 then successive terms always decrease and $E_n \to 0$ as $n \to \infty$

$$|E_{n+1}| = \frac{|x|^{n+2}}{(n+2)!} \qquad |E_n| = \frac{|x|^{n+1}}{(n+1)!} \qquad \frac{|E_{n+1}|}{|E_n|} = \frac{|x|^{n+2}}{|x|^{n+1}} \cdot \frac{(n+1)!}{(n+2)!}$$

$$= \frac{|x|}{(n+2)}$$

but $n = 0, 1, 2, 3 \ldots$ so the error is always decreasing

and $|E_n| \to 0$ as $n \to \infty$

we really can write $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

YOU TRY: Bound the error in our approximation of $\sin(.1)$ using P_3.

$$|E_3| \le \frac{M}{4!}|x|^4 \qquad M \ge f^{(4)}(x) = \frac{d^4}{dx^4}\cos(x) = \sin(x)$$

so $M=1$ would work.

$$|E_3| < \frac{|x|^4}{4!} = \frac{|.1|^4}{24} = \frac{.00001}{24} = 4.16 \times 10^{-6}$$

we should be correct out to 5 digits.

$$\sin(.1) = 0.09983333$$

$\uparrow$ good.

We could also ask how many terms would we need to keep to get accuracy 1×10^{-10}

$$|E_n| \le \frac{M}{(n+1)!}|x|^{n+1} < \frac{1}{(n+1)!}|x|^{n+1} = \frac{|.1|^{n+1}}{(n+1)!} \underset{\text{want}}{<} 1 \times 10^{-10}$$

use a computer or plug in n-values.

$n=5 \quad E_n \le 1.388 \times 10^{-9}$

when $n=6 \quad E_n \le 1.984 \times 10^{-11}$

so we would need $P_7(x)$ since $\sin(x)$ only has odd terms in the Taylor Series!