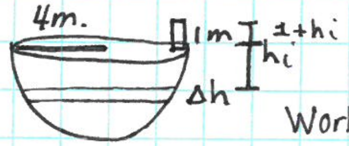
Review – Integration and Applications.

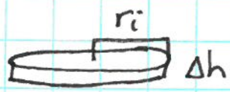
Ex Find the work done in removing water from the hemispherical bowl to a point 1 m above the rim.

#1



Work = Force × Distance   Density of water = 1000 kg/m³

#2 Slice of constant WORK



Work = Force × Distance

Force $(kg\,m/s^2)$ = Density • Volume • Gravity

Distance → h_i+1

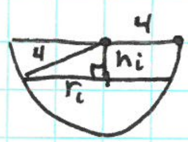
$1000\ kg/m^3 \cdot \pi r_i^2 \Delta h\ m^3 \cdot 9.8\ m/s^2$

$9800\pi r_i^2 \Delta h\ kg \cdot m/s^2$ ✓

$$W_i = 9800\pi\, r_i^2 (h_i+1)\Delta h$$

#3 need r_i in terms of h_i

"radius changes based on height".



Round – Circular – Pythagorean

$r_i^2 + h_i^2 = 16 \qquad r_i = \sqrt{16-h_i^2}$

$$W_i = 9800\pi(16-h_i^2)(h_i+1)\Delta h.$$

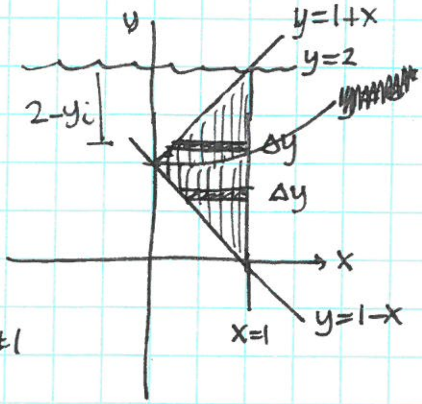
#4/5 Add up the slices ⟶ Integrate   Total Work $= \lim_{n\to\infty} \sum_{i=1}^{n} 9800\pi(16-h_i^2)(h_i+1)\Delta h.$

$$\text{Work} = 9800\pi \int_0^4 16h - h^3 + 16 - h^2\, dh$$

$$= 9800\pi\left[4h^2 - \frac{h^4}{4} + 16h - \frac{h^3}{3}\right]\Big|_0^4 = 9800\pi\left[4(4)^2 - \frac{4^4}{4} + 16(4) - \frac{4^3}{3}\right]$$

$$= 9800\pi\left[64 - \frac{64}{3}\right] = 9800\pi\, \frac{2 \cdot 64}{3}$$

$$= \boxed{9800\pi\, \frac{128}{3}\ N}$$



#1

Ex Find the Fluid Force on a plate bounded by $y = 1+x$, $y = 1-x$, and $x=1$ if the surface of the water is at $y=2$. (assume meters)

Fluid Force = Pressure x Area.

Pressure = Density$_{\text{water}}$ · gravity · depth.

#2 Find the Fluid Force on a Slice
"Slice at constant depth"

TWO DIFFERENT REGIONS !!

- Solve them separately then add them up.

TOP 1/2

w_i ▭ Δy $\quad P_i = (1000)(9.8)(2-y_i)$

$$F_i = (1000)(9.8)(2-y_i)\underbrace{w_i\,\Delta y}_{\text{area of the slice}}$$

#3 Get w_i in terms of y

$$w_i = x_2 - x_1 = 1 - (y_i - 1) = 2 - y_i$$

line $x=1$ on the right; line $y=1+x$ or $x=y-1$ on the left.

$$F_i = 9800(2-y_i)(2-y_i)\,\Delta y$$

BOTTOM 1/2

w_i ▭ Δy $\quad P_i = (1000)(9.8)(2-y_i)$

$$F_i = 9800(2-y_i)\,w_i\,\Delta y$$

$$w_i = x_2 - x_1 = 1 - (1-y) = y_i$$

now we have a different LEFT SIDE!

$$F_i = 9800(2-y_i)\,y_i\,\Delta y$$

#4/5
INTEGRATE (Add up the pieces)

$$\text{TOTAL FORCE} = \text{TOP FORCE} + \text{BOTTOM FORCE} = 9800\int_1^2 4 - 4y + y^2\,dy + 9800\int_0^1 2y - y^2\,dy$$

$$= 9800\left[\left(4y - 2y^2 + \frac{y^3}{3}\right)\Big|_1^2 + \left(y^2 - \frac{y^3}{3}\right)\Big|_0^1\right]$$

$$= 9800\left[\left(8 - 8 + \frac{8}{3}\right) - \left(4 - 2 + 1/3\right) + \left(1 - 1/3\right)\right] = 9800\left[\frac{8}{3} - 7/3 + 2/3\right] = \boxed{9800\ \text{N}}$$

Integrate: $\int \frac{1}{x^2+3x+2}\,dx \longrightarrow \int \frac{1}{(x+1)(x+2)}\,dx$

Polynomial in Denom
Try to factor

Now do
partial Fractions.

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x+1)$$

$$1 = (A+B)x + (2A+B)$$

$A+B=0$ and $2A+B=1$

$A=-B$ $\quad -2B+B=1$

$B=-1$

$A=1$

so

$$\int \frac{1}{(x+1)(x+2)}\,dx = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx = \ln|x+1| - \ln|x+2| + C$$

Integrate: $\int \frac{\sin t}{\cos^3 t}\,dt \longrightarrow \int \frac{\sin t}{u^3}\frac{du}{\sin t} = \int \frac{du}{u^3} = \int u^{-3}\,du$

do a u-sub!

$u = \cos t$

$du = \sin t\,dt$

$dt = \frac{du}{\sin t}$

$$= \frac{-u^{-2}}{2} + C = \frac{-1}{2u^2} + C = \frac{-1}{2\cos^2 t} + C$$

Integrate: $\int x\ln(x)\,dx$

Formula $\int u\,dv = uv - \int v\,du$

From the product rule

$\ln(x) \rightarrow$ integration by parts.

always choose $u=\ln(x)$ let $u=\ln(x)$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

Integrate: $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{x-1}}\,dx$

This is improper!
When $x=1$ the integrand is undefined!

Consider $\displaystyle\int_{a}^{2} \frac{1}{\sqrt{x-1}}\,dx$

let $u = x-1$
$du = dx$
$x=a \quad u=a-1$
$x=2 \quad u=1$

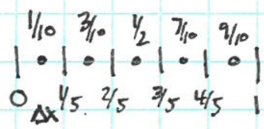
$$\longrightarrow \int_{a-1}^{1} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{a-1}^{1} = 2-2\sqrt{a-1}$$

$$\lim_{a\to 1} 2-2\sqrt{a-1} = 2$$

Ex Numerical Integration

Use MID(5) to integrate $\displaystyle\int_0^1 e^{x^2}\,dx$.

I need 5 midpoints between 0 an 1



height of boxes determind by a point in middle.

$\Delta x = 1/5$

$$\int_0^1 e^{x^2}\,dx \sim f(1/10)\Delta x + f(3/10)\Delta x + f(1/2)\Delta x + f(7/10)\Delta x + f(9/10)\Delta x$$

on a calculator...

$$= (1.010)1/5 + (1.094)1/5 + (1.284)1/5 + (1.632)1/5 + (2.248)1/5$$

$$= 1.454$$

Ex Compare this with using T_4 to approximate $f(x) = e^{x^2}$ in the integral.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \text{so} \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$$

here T_4 is the fourth order $\quad T_4 = 1 + x^2 + \dfrac{x^4}{2!}$

Use this in the integral.

$$\int_0^1 1 + x^2 + \frac{x^4}{2!}\, dx = x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!}\Big|_0^1 = 1 + 1/3 + 1/10 = 1.4333.$$

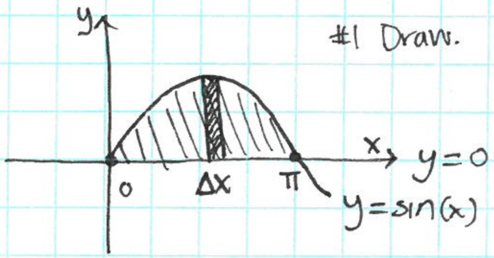
"Real": $\int_0^1 e^{x^2} dx \approx 1.46265$

Both are accurate to the first decimal place.

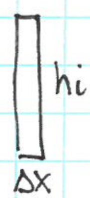
<u>Ex</u> Find the mass of the region bounded by $y = \sin(x)$ and $y = 0$ between $0 \le x \le \pi$ if the density is given by $y = e^x$.

assume units are kg/m^2

#1 Draw.



#2 Find a slice of constant density.



<u>Mass</u> of <u>Slice</u>

$M_i = \text{density} \times \text{area}$
$= e^{x_i} \cdot h_i \, \Delta x$

#3 Need h_i in terms of x

the height depends on x ... $h_i = \sin(x_i)$

$M_i = e^{x_i} \sin(x_i)\, \Delta x.$

#4/5 Add up the Slices, and Integrate.

$$\text{Mass} = \int_0^{\pi} e^x \sin(x)\, dx$$

Integrate by parts twice!

First do Integral

$$\int e^x \sin(x)\, dx = \sin(x) e^x - \int e^x \cos(x)\, dx = \sin(x) e^x - \cos(x) e^x - \underbrace{\int e^x \sin(x)\, dx}_{\text{copy.}}$$

let $u = \sin(x)$ $dv = e^x dx$
$du = \cos(x) dx$ $v = e^x$

let $u = \cos(x)$ $dv = e^x dx$
$du = -\sin(x) dx$ $v = e^x$

Solve algebraically.

bring the copy to the LHS

$$2\int e^x \sin(x)\, dx = \sin(x) e^x - \cos(x) e^x$$

or $$\int e^x \sin(x)\, dx = \frac{1}{2}\left[\sin(x) - \cos(x)\right] e^x + C$$

Now use this information to solve the problem

$$\text{Mass} = \int_0^{\pi} e^x \sin(x)\,dx = \frac{1}{2}e^x\left(\sin(x) - \cos(x)\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\overset{0}{\cancel{\sin(\pi)}} - \overset{-1}{\cancel{\cos(\pi)}}\right)e^{\pi} - \frac{1}{2}\left(\overset{0}{\cancel{\sin(0)}} - \overset{1}{\cancel{\cos(0)}}\right)e^{0}$$

$$= \boxed{\frac{1}{2}e^{\pi} + 1 \ \text{kg.}}$$