Review of Sequences and Series.

Given $\sum_{n=0}^{\infty} a_n$

does $\lim_{n\to\infty} a_n = 0$?

NO → The series diverges. DIVERGENCE TEST

YES

Is it geometric $\sum_{n=0}^{\infty} a(x)^n$

YES → Use what we know about geometric series. Converges if $|x|<1$

NO

Are all of the terms (a_n) positive? → YES → can I easily integrate it? → YES

Integral test $\int_0^{\infty} a_n\,dn$

NO

Are $a_n > 0$ and do I see factorials or exponents → YES

Ratio test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

NO

Use comparison or Limit comparison.

NO

Is it alternating $(-1)^n$

NO

YES

Do the alternating series (Liebniz) test

$\lim_{n\to\infty} a_n = 0$ converges at least conditionally.

Compare to p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+2n}$

The terms are going to zero: $\lim_{n\to\infty} \frac{1}{\sqrt{n}+2n} = 0$

and the series is alternating so it at least converges conditionally.

Consider the absolute value:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}+2n} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2n}$$

Try comparison or limit comparison.

Behaves as $b_n = 1/n$ as $n \to \infty$

Limit compare:

$$\frac{a_n}{b_n} = \frac{1/\sqrt{n}+2n}{1/n} = \frac{n}{2n+\sqrt{n}} = \frac{n \cdot 1}{n(2+n^{-1/2})} = \frac{1}{(2+n^{-1/2})}$$

$$\lim_{n\to\infty} \frac{1}{(2+n^{-1/2})} = \frac{1}{2}$$

since this limit is finite Σa_n and Σb_n either BOTH converge or BOTH diverge.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges — p-series with $p=1$

so $\sum_{n=1}^{\infty} \frac{1}{2n+\sqrt{n}}$ diverges.

<u>and</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+\sqrt{n}}$ converges conditionally.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^8}$ Use the Ratio Test.

Ratio Test $a_n = \dfrac{\sqrt{n+1}}{n^8}$ $a_{n+1} = \dfrac{\sqrt{n+2}}{(n+1)^8}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+2}}{\sqrt{n+1}}\cdot\frac{n^8}{(n+1)^8}\right| = \lim_{n\to\infty}\left|\sqrt{\frac{n+2}{n+1}}\cdot\left(\frac{n}{n+1}\right)^8\right| = 1$$

so the ratio test is inconclusive.

Ex Find the interval of convergence. — for what values of x does this series converge.

$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n+1}$ Use RATIO TEST.

$a_n = \dfrac{x^n}{n+1}$ $a_{n+1} = \dfrac{x^{n+1}}{n+2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n+1}{n+2}\right| = \lim_{n\to\infty}\left|x\cdot\frac{n+1}{n+2}\right| = |x|$$

$|x| < 1$ for convergence so $R = 1$

Interval of convergence $\underset{?}{-1 <} x \underset{?}{<} 1$

Endpoints

$x = -1$ then $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ this converges by the alternating series test: $\displaystyle\lim_{n\to\infty}\frac{1}{n+1} = 0$

$x = 1$ then $\displaystyle\sum_{n=0}^{\infty}\frac{1^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ behaves like $b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/n+1}{1/n} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

since $\sum 1/n$ diverges so does $\sum \frac{1}{n+1}$

Interval of convergence $-1 \le x < 1$

Ex Write $\frac{1}{3+9x}$ as an infinite series by comparing to $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$f(x) = \frac{1}{3+9x} = \frac{1}{3}\left(\frac{1}{1+3x}\right) = \frac{1}{3}\left(\frac{1}{1-(-3x)}\right) = \frac{1}{3}\left(\frac{1}{1-u}\right) = \frac{1}{3}\sum_{n=0}^{\infty} u^n$$

let $u = -3x$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-3x)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n 3^n x^n = \sum_{n=0}^{\infty}(-1)^n 3^{n-1} x^n$$

converges when $|-3x| < 1$

$|x| < 1/3$

$-1/3 < x < 1/3$

Ex

Find n so that the error in using $T_n(x)$ to approximate $f(x) = e^x$ centered at $x=0$ is less than

$|E_n| < 10^{-8}$ on $0 < x < 2$

Use the formula $|E_n| \le \frac{\overbrace{f^{(n+1)}(c)}^{=M}}{(n+1)!}|x-a|^{n+1}$

$|E_n| \le \frac{M}{(n+1)!}|x|^{n+1}$ $\quad f^{(n+1)}(c) = e^c$ where $0 \le c \le 2$

how bad could it get?

$e^2 < 9 = M$

$|E_n| \le \frac{9}{(n+1)!}|2|^{n+1} < 10^{-8}$

on a calculator.

$n=12$ $\quad E_{12} \sim 10^{-5}$

$n=14$ $\quad E_{14} \sim 10^{-7}$

$n=15$ $\quad E_{15} \sim 10^{-8}$

$n=16$ $\quad E_{16} \sim 10^{-9}$ ✓

We would need $T_{16}(x)$ to approximate $f(x) = e^x$ with 10^{-8} accuracy on $0 \le x \le 2$.