TODAY.
Trigonometric Identities and Trig Substitutions.

FIRST... where are we...

1. Integrals we know... "TABLE FORM" Ex. $\int x^2 dx$ $\int \sin x dx$ ones we just know.
2. Simple algebra ... Ex $\int \frac{x+1}{x} dx$ $\int (x^2+3)\sqrt{x}\, dx$
3. Substitution.. Ex. $\int x\sin(x^2)\,dx$ $\int \frac{x^2}{\sqrt{x^3+3}}\,dx$
   Function inside a function.
4. Integration by parts Ex. $\int x^2\sin(x)\,dx$ $\int xe^x dx$ $\int e^x \sin x\, dx$
   Product of two functions... no inside or sub wont work.
5. Algebraic Simplification
   A. Partial Fractions – order of Denom > order of Num
   B. Long Division – order of Num ≥ order of Denom
   C. Complete the Square – order of Denom > order of Num AND can't factor nicely.
6. Trigonometric Identities and Trig Sub.

JUST KNOW THESE:

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x + \cos^2 x = 1$

ODD POWERS
factor one out
use $\sin^2+\cos^2=1$, usub

$\int \sin^n x\, dx$ or $\int \cos^n x\, dx$

EVEN POWERS
separate into even factors
use double angles.

Ex. $\int \sin^4 x \cos^2 x\, dx$

Here we want all sines or all cosines,
replace the $\cos^2 x = 1-\sin^2 x$

$= \int \sin^4 x(1-\sin^2 x)\,dx = \int -\sin^6 x + \sin^4 x\, dx$

then use even odd

For many of these – in real life we will use a table of integrals.

Ex $\int \sin 2x \cos 4x\, dx = -\frac{\cos(-2x)}{2(-2)} - \frac{\cos(6x)}{2(6)} + C.$

Rarely use a table.

#24 on table
m=2 and n=4

why $m \neq n$?

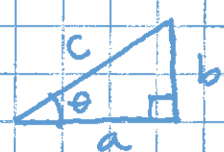
[redacted]

Last Time: Using algebra to simplify the integrand
— this is almost always helpful to do.

Today [redacted] Trigonometric Substitutions.

• use when you see terms like:
$\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ $\dfrac{1}{(a^2+x^2)^n}$ ....

Pathagorean Theorem: $a^2+b^2=c^2$

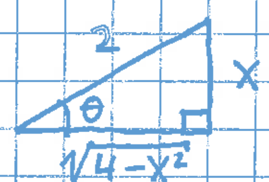


and $\sin\theta = \dfrac{b}{c}$ $\cos\theta = \dfrac{a}{c}$ $\tan\theta = \dfrac{b}{a}$

Use these ideas to our advantage.

Ex. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ • looks like Pathagorean Thm.
$a^2 = c^2 - b^2$
or $a^2 = 4-x^2$ so $a=\sqrt{4-x^2}$
$c=2$ $b=x$

1. choose a, b and c
the write down triangle



always put x opposite!

2. Then read things off triangle: $\sin\theta = \dfrac{x}{2}$

$\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

so the easiest substitution is: $x = 2\sin\theta$

then $dx = 2\cos\theta \, d\theta$

and $\frac{1}{\sqrt{4-x^2}} = \frac{1}{2\cos\theta}$

3. Sub it all in: $\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{2\cos\theta} \cdot 2\cos\theta \, d\theta = \int d\theta$

4. Solve the easier integral...

$\int d\theta = \theta + C$ undo the substitution

$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$ $\quad \theta = \arcsin(\frac{x}{2})$

$= \boxed{\arcsin(\frac{x}{2}) + C}$

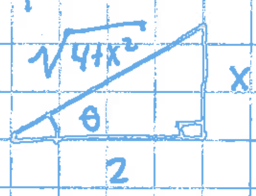
Ex $\int \frac{1}{(4+x^2)} dx$ • looks like pathagorean thm in denom

$\underbrace{a^2+b^2}_{} = c^2$

1. Choose a, b, and c. $4 + x^2 = c^2$

$a = 2 \quad b = x$ and $c = \sqrt{4+x^2}$

put it all on a triangle



→ always put x opposite.

2. Read things off the triangle...

$\tan\theta = \frac{x}{2} \qquad \cos\theta = \frac{2}{\sqrt{4+x^2}}$

easiest sub: $x = 2\tan\theta$

then $dx = \frac{2}{\cos^2\theta} d\theta$

and $\frac{1}{4+x^2} = \frac{1}{4}\cos^2\theta$

3. Plug everything in:

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4}\cos^2\theta \cdot \frac{2}{\cos^2\theta}\,d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

4. Solve and sub back.

$$\tan\theta = \frac{x}{2} \quad \text{so} \quad \theta = \arctan\left(\frac{x}{2}\right)$$

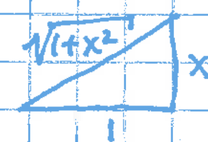
$$= \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

You Try: #2 $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$

hint... remember when integrating even powers of sine or cosine $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$x = \tan\theta$

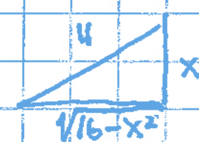
$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$



$\theta = \arctan(x)$

$$= \frac{1}{2}\arctan(x) + \frac{1}{4}\sin[2\arctan(x)] + C$$

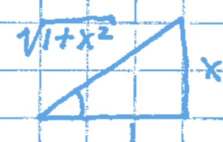
#1 $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$ $\quad x = 4\sin\theta$



$$= \arcsin\left(\frac{x}{4}\right) + C$$

Do This One in Class.

★★ #3 $\displaystyle\int \frac{1}{(1+x^2)^{5/2}}\,dx$ $\quad x = \tan\theta$



$$= \int \cos^3\theta\,d\theta = \sin\theta - \frac{1}{3}\sin^3\theta + C$$

$$\sin\theta = \frac{x}{(1+x^2)^{1/2}} \quad \text{and} \quad \sin^3\theta = \frac{x^3}{(1+x^2)^{3/2}}$$

$$= \frac{x}{(1+x^2)^{1/2}} - \frac{1}{3}\frac{x^3}{(1+x^2)^{3/2}} + C$$

Then do the integration worksheet !!

#1. look at each one... what method and why?

HINTS FOR INTEGRATION.

IN GENERAL... when choosing a method of integration I choose in the following order:

1. Think about it ... do I already know the answer?

   x^n, $\sin x$, $\cos x$, e^x ... $\frac{1}{\sqrt{1-x^2}}$, $\frac{1}{x^2+1}$, ... $\frac{1}{x}$

   Flashcards ... memorize these.

2. Is there some simple algebra or cancelation?

   $$\frac{x^2+x}{x} = x+1 \qquad \tan\theta \cdot \cos\theta = \sin\theta$$

3. Try a substitution ... Is there is a function inside a function?

   xe^{x^2}, $x^2\sin(x^3)$, $x^2\sqrt{x^3+1}$, $\sin^2\theta\cos\theta$

   NOTE ... $\int \sec\theta \, d\theta$ let $u = \sec\theta + \tan\theta$

   there are some tricky ones.

   $\int \tan\theta \, d\theta$ let $u = \cos\theta$.

4. Integration by parts ... Is there a product of functions?

   xe^x, $x^2\sin x$, $x\ln(x)$, $e^x\cos x$

5. Algebraic Simplification ...
   - Partial Fractions. DENOM > NUM + FACTOR
   - Long Division. NUM > DENOM
   - Complete the square. DENOM > NUM can't FACTOR.

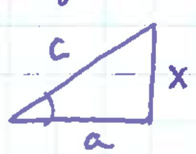
6. Trigonometric Identities.

   $\sin^2 x + \cos^2 x = 1$

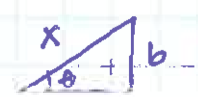
   $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

   $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

7. Trigonometric Substitution ... Do I see terms like a^2+b^2 or c^2-b^2, that can't be simplified.



only in special case.



EXTRA EXAMPLES — TABLE INTEGRALS.

Sometimes for table integrals we will need to do a substitution first.

Recall: from 5.7 $\int \frac{1}{x^2+1}\,dx = \arctan(x)+c$ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)+c$

Ex. $\int \frac{1}{\sqrt{9-16x^2}}\,dx$

we need $\frac{1}{\sqrt{9-16x^2}} \xrightarrow[\text{like}]{\text{look}} \frac{1}{\sqrt{1-x^2}}$

transform to something we know.

Factor out the 9:

$$\frac{1}{\sqrt{9-16x^2}} = \frac{1}{\sqrt{9\left(1-\frac{16x^2}{9}\right)}} = \frac{1}{3\sqrt{1-\frac{16x^2}{9}}}$$

now let $u^2 = \frac{16x^2}{9} = \left(\frac{4x}{3}\right)^2$ in other words $u = \frac{4x}{3}$

then $du = \frac{4}{3}\,dx$

so $$\int \frac{1}{\sqrt{9-16x^2}}\,dx = \int \frac{1}{3}\frac{1}{\sqrt{1-\frac{16x^2}{9}}}\,dx = \int \frac{1}{3}\frac{1}{\sqrt{1-u^2}}\cdot\frac{3}{4}\,du$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{4}\arcsin(u)+c$$

$$= \frac{1}{4}\arcsin\left(\frac{4x}{3}\right)+c.$$

YOU TRY. $\int \frac{1}{4t^2+9}\,dt = \frac{1}{6}\arctan\left(\frac{2t}{3}\right)+c$

$\int \frac{1}{9}\cdot\frac{1}{\frac{4t^2}{9}+1}\,dt$ let $u^2 = \frac{4t^2}{9}$ or $u = \frac{2t}{3}$

This works when things factor nicely... lets see where all of this comes from, so we can deal w/ the not so nice cases.

EXTRA EXAMPLES — SECANT and TRIG SUB

TODAY. Integrals involving secant:

From the homework... a special case of substitution:

EX.

$$\int \sec x \, dx$$

let $u = \sec x + \tan x$

then $du = (\sec x \tan x + \sec^2 x)\,dx$

$$dx = \frac{du}{\sec x(\tan x + \sec x)} = \frac{du}{u \sec x}$$

$$\int \sec x \cdot \frac{du}{u \sec x} = \int \frac{1}{u}\,du = \ln|u| + c$$

$$= \ln|\sec x + \tan x| + c$$

A special case of trigonometric sub... #39.

EX.

$$\int \frac{dx}{\sqrt{x + 6x^2}}$$

★ factor out the 6 : $6(x^2 + \frac{1}{6}x)$

complete the square:

$$\frac{1}{6}x + x^2 = (x+b)^2 + c$$
$$= x^2 + 2bx + b^2 + c$$

$2b = 1/6$ and $1/144 + c = 0$

$b = 1/12$ and $c = -1/144$

$$x^2 + \tfrac{1}{6}x = (x + 1/12)^2 - 1/144$$

$$\int \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{(x+1/12)^2 - 1/144}}\,dx$$

★ now do a substitution.

let $u = x + 1/12$

$du = dx$

$$= \int \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{u^2 - 1/144}}\,dx$$

★ now do a trig sub!

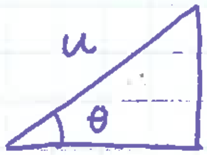
NOTE ... The only time we write our triangle differently is when we have $(\text{var})^2$ — constant

continued on next page.

normally a^2-x^2

this is a strange case! <u>our variable is the hypotenuse!</u>

#1 Construct the triangle:



put the stuff w/ minus on the side across from θ

#2 read the most simple angle ... secant.

$\cos\theta = \frac{1/12}{u}$ or $u = \frac{1}{12}\sec\theta$

leads to a secant substitution

#3 find du then $du = \frac{1}{12}[\sec\theta \tan\theta]\,d\theta$

#4 read tangent also $\tan\theta = \frac{\sqrt{u^2-1/144}}{1/12}$

$$\frac{1}{\sqrt{6}}\int \frac{1}{\sqrt{u^2-1/144}}\,du = \frac{1}{\sqrt{6}}\int \frac{12}{\tan\theta}\cdot\frac{1}{12}[\sec\theta\tan\theta]\,d\theta$$

$$= \frac{1}{\sqrt{6}}\int \frac{\sec\theta}{\cancel{\tan\theta}}\,d\theta = \frac{1}{12\sqrt{6}}\int \sec\theta\,d\theta$$

$$= \frac{1}{\sqrt{6}}\left[\ln|\sec\theta+\tan\theta|\right]+c$$

Now: $\sec\theta = 12u$ and $\tan\theta = \frac{\sqrt{u^2-1/144}}{1/12}$

$$= \frac{1}{\sqrt{6}}\left[\ln\left|12u+12\sqrt{u^2-1/144}\right|\right]+c$$

rewrite in terms of u

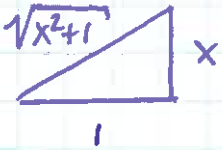
$$\int\frac{dx}{\sqrt{x+6x^2}} = \frac{1}{\sqrt{6}}\left[\ln\left|12x+1+12\sqrt{(x+1/12)^2-1/144}\right|\right]+c$$

$$= \frac{1}{\sqrt{6}}\left[\ln\left|12x+1+12\sqrt{x^2+1/6x}\right|\right]+c$$

rewrite in terms of x.

Another Example of Trig Sub:

Ex $\displaystyle\int \frac{dx}{(x^2+1)^3}$



Then $\tan\theta = x$

$x = \tan\theta$

$dx = \dfrac{d\theta}{\cos^2\theta}$ $\longrightarrow$ $\displaystyle\int \frac{\cos^6\theta}{\cos^2\theta}\, d\theta$

$\cos\theta = \dfrac{1}{\sqrt{x^2+1}}$

and $\cos^6\theta = \dfrac{1}{((x^2+1)^{1/2})^6} = \dfrac{1}{(x^2+1)^3}$

$$\int \cos^4\theta \, d\theta = \frac{1}{4}\int 1 - 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\int 1 - 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\, d\theta$$