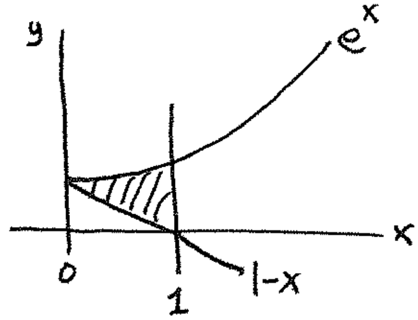
1. Find the area of the region enclosed by $y = e^x$, $y = 1 - x$, and $x = 1$



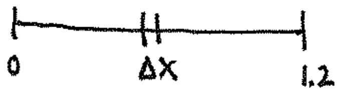
$$A = \int_0^1 e^x - (1-x)\,dx$$

$$= \int_0^1 x + e^x - 1\,dx$$

$$= \frac{x^2}{2} + e^x - x \Big|_0^1$$

$$= \left[\frac{1}{2} + e - 1\right] - [1] = \boxed{e - 3/2}$$

2. Find the total mass of a rod of length 1.2m with linear density $\rho(x) = (1 + 2x + \frac{2}{9}x^3)$ kg/m.



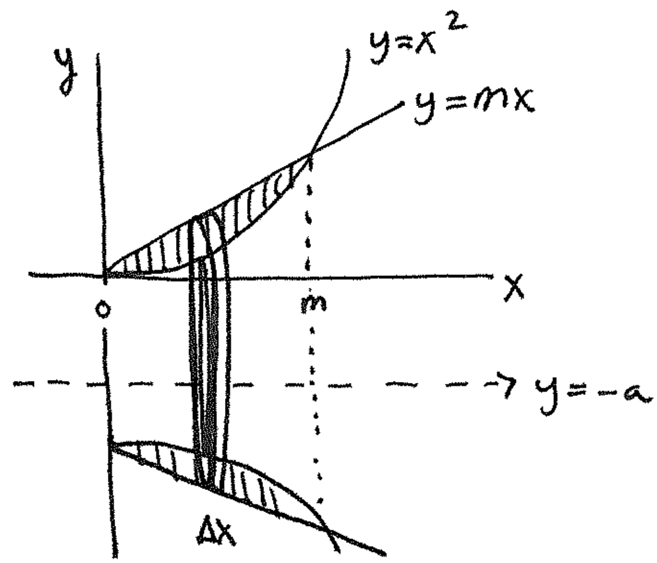
$\text{mass}_i = \rho(x_i)\Delta x$

$$\text{total mass} = \int_0^{1.2} 1 + 2x + \frac{2}{9}x^3 \, dx$$

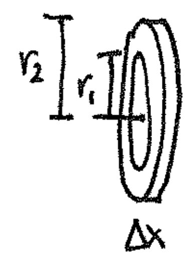
$$= x + x^2 + \frac{2}{27}x^4 \Big|_0^{1.2}$$

$$= \boxed{1.2 + (1.2)^2 + \frac{2}{27}(1.2)^4}$$

3. Find the volume obtained by rotating the region bounded by the parabola $y = x^2$ and the line $y = mx$ about the axis $y = -a$. Where m and a are positive numbers with m representing the slope of the line.



Washer Method



$$V_i = \pi(r_2^2 - r_1^2)\Delta x$$

$$r_2 = mx_i + a$$

$$r_1 = x_i^2 + a$$

$$V_i = \pi\left((mx_i+a)^2 - (x_i^2+a)^2\right)\Delta x$$

$$= \pi\left[m^2x_i^2 + 2mx_i + \cancel{a^2} - x_i^4 - 2ax_i^2 - \cancel{a^2}\right]\Delta x$$

$$= \pi\left[(m^2-2a)x_i^2 - x_i^4 + 2mx_i\right]\Delta x$$

Add up the pieces:

$$\text{Total Volume} = \pi\int_0^m (m^2-2a)x^2 - x^4 + 2mx \, dx$$

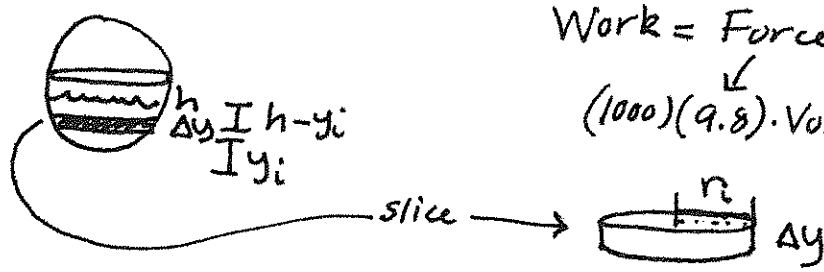
$$= \pi\left[\frac{(m^2-2a)x^3}{3} - \frac{x^5}{5} + mx^2\right]\Bigg|_0^m$$

$$= \boxed{\pi(m^2-2a)\frac{m^3}{3} - \frac{\pi m^5}{5} + \pi m^3}$$

4. Water is pumped INTO a spherical tank of radius 2m from the bottom. The density of water is 1000 kg/m^3. How much work is needed to fill the tank to a level of h meters?

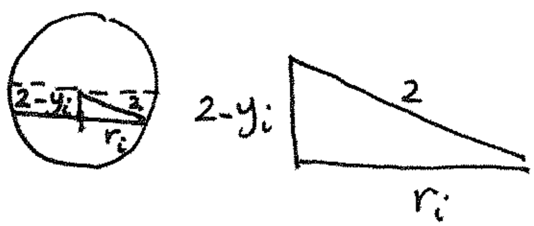
Work = Force × Distance

$(1000)(9.8)\cdot\text{Volume}$ $\qquad y_i$



$$W_i = 9800\pi r_i^2 y_i \Delta y$$

need r_i in terms of y_i



$$r_i^2 = 4 - (2-y_i)^2$$

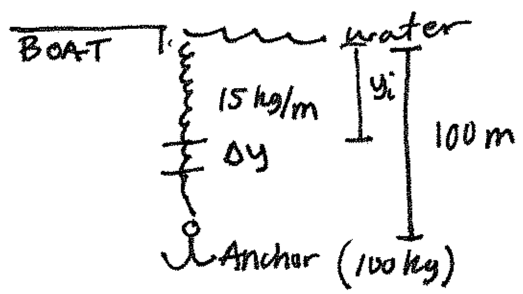
$$W_i = 9800\pi\left(4-(2-y_i)^2\right)y_i\,\Delta y$$

Add up slices:

$$\text{Total Work} = 9800\pi\int_0^h 4y - (2-y)^2 y \, dy = 9800\pi\int_0^h \cancel{4y} - (\cancel{4} - 4y + y^2)y \, dy$$

$$= 9800\pi\int_0^h -y^3 + 4y^2 \, dy = \boxed{9800\pi\left[-\frac{h^4}{4} + \frac{4}{3}h^3\right]}$$

5. A 100 kg anchor is attached to the bottom of a chain whose density is 15 kg/m. How much work is done in raising the chain from being fully extended, 100 meters up to the deck of the boat. (assume buoyancy has no effect)



TWO PROBLEMS:

#1 ANCHOR: WORK = F × d

$= (100)(9.8)(100)$

$= 98{,}000$ Nm

#2 CHAIN: need calculus.

I Δy WORK = F × d

$F \to (15)(9.8)\Delta y$, $d \to y_i$

$W_i = (15)(9.8)\, y_i\, \Delta y$

TOTAL WORK CHAIN $= (15)(9.8) \int_0^{100} y\, dy = (15)(9.8)\dfrac{(100)^2}{2}$

TOTAL WORK $= (15)(9.8)\dfrac{(100)^2}{2} + 98000$ NM.

units Newton-meters.

= JOULE. (J)

add them up

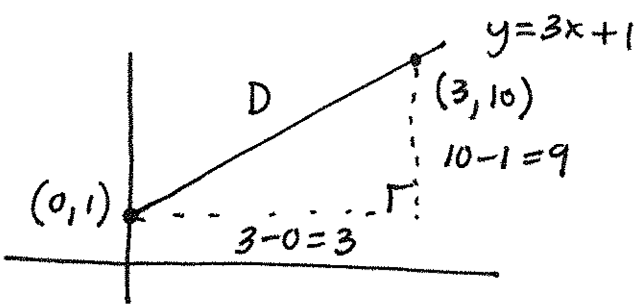
$\boxed{W = 833{,}000 \text{ J}}$

6. Calculate the arc length of $y = 3x + 1$ over the interval $[0, 3]$. First use integration, then check your answer using the distance formula.

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f'(x) = 3$$

$$\text{Arc Length} = \int_0^3 \sqrt{1+9}\, dx = \sqrt{10} \int_0^3 dx = \boxed{3\sqrt{10}} \checkmark$$



$$D = \sqrt{3^2 + 9^2}$$
$$= \sqrt{90} = \sqrt{9 \cdot 10} = 3\sqrt{10} \checkmark$$

7. Calculate the arc length of $y = x^{\frac{3}{2}}$ over the interval $[1, 2]$.

$$f'(x) = \frac{3}{2}x^{1/2}$$

$$\text{Arc length} = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

let $u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}dx \qquad dx = \frac{4}{9}du$

$x = 1 \longrightarrow u = \frac{13}{4}$

$x = 2 \longrightarrow u = \frac{22}{4}$

$$\text{Arc length} = \frac{4}{9}\int_{13/4}^{22/4} \sqrt{u}\, du = \frac{8}{27} \cdot u^{3/2} \Big|_{13/4}^{22/4}$$

$$= \boxed{\frac{8}{27}\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]}$$

8. Find the surface area of revolution for the region bounded by the line $y = 4x + 3$ between $[0, 1]$ and $y = 0$ rotated about the x-axis.

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

$$f'(x) = 4$$

$$\text{Surface Area} = 2\pi \int_0^1 (4x+3)\sqrt{1+16}\,dx = 2\pi\sqrt{17}\int_0^1 4x+3\,dx$$

$$= 2\pi\sqrt{17}\left[2x^2 + 3x\right]\Big|_0^1 = 2\pi\sqrt{17}\left[2+3\right]$$

$$= \boxed{10\pi\sqrt{17}}$$

9. Derive the formula for the surface area of revolution if you rotate the curve $y = f(x)$ about the line $y = -a$. NOTE: Assume that your region goes all the way to the axis of rotation. In other words, your volume is NOT hollow.

the only difference is the distance to the axis of rotation.

Same as derivation in class...

Surface Area = circumference × length.

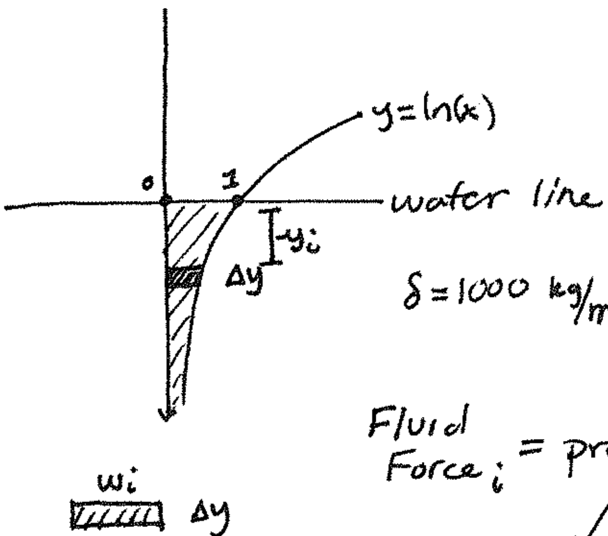
$2\pi r_i$ $\quad\quad$ $\sqrt{1+f'(x)^2}\,\Delta x$

now $r_i = f(x_i)+a$

So

$$\boxed{\text{Surface Area} = 2\pi \int_a^b [f(x)+a]\sqrt{1+f'(x)^2}\,dx}$$

10. Find the fluid force on the "infinite" plate bounded by the curve $y = ln(x)$ for $0 \leq x \leq 1$. NOTE: This will result in an improper integral!



$\delta = 1000\ \text{kg}/\text{m}^3$

Fluid Force$_i$ = pressure × area

pressure → $(1000)(9.8)$ depth, depth → $-y_i$; area → $w_i \Delta y$

$= -9800\, y_i w_i \Delta y$

need w_i in terms of y_i ... $y_i = \ln(x)$

$x = e^{y_i}$

$w_i = e^{y_i} - 0 = e^{y_i}$

$F_i = -9800\, y_i e^{y_i} \Delta y$

TOTAL FORCE $= -9800 \int_{-\infty}^{0} y e^{y}\, dy$

$\int_b^0 y e^y\, dy = y e^y \Big|_b^0 - \int_b^0 e^y\, dy$

let $u = y$ $dv = e^y dy$

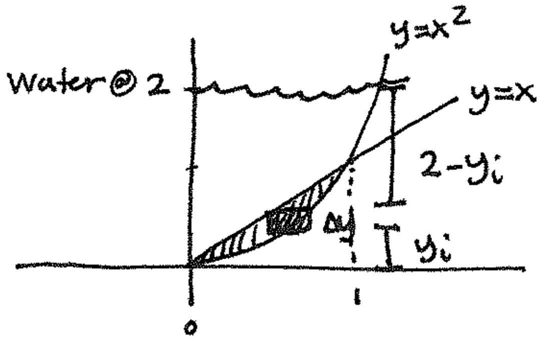
$du = dy$ $v = e^y$

$= y e^y - e^y \Big|_b^0 = -e^0 - \left[b e^b - e^b \right]$

$\lim_{b \to -\infty} -1 - b e^b + e^b = -1$

TOTAL FORCE $= -9800(-1) = \boxed{9800\ N}$

11. Find the fluid force on the plate bounded by $y = x^2$ and $y = x$, if the plate is submerged underwater with the water line at $y = 2$.



Consider a slice @ constant depth:

w_i Δy

F_i = Pressure × Area

Pressure: $(1000)(9.8)$depth, depth $= 2-y_i$

Area: $w_i \Delta y$

need w_i in terms of y_i:

$w_i = x_{RIGHT} - x_{LEFT}$
$= \sqrt{y_i} - y_i$

$F_i = 9800(2-y_i)(\sqrt{y_i} - y_i)\Delta y$

$$\text{TOTAL FORCE} = 9800\int_0^1 (2-y)(y^{1/2}-y)\,dy = 9800\int_0^1 2y^{1/2} - y^{3/2} - 2y + y^2\,dy$$

$$= 9800\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2} - y^2 + \frac{y^3}{3}\right]\Bigg|_0^1$$

$$= 9800\left[\frac{4}{3} - \frac{2}{5} - 1 + \frac{1}{3}\right] = 9800\left[\frac{5}{3} - \frac{7}{5}\right]$$

$$= \boxed{9800\left[\frac{4}{15}\right]}$$

simplify to make sure the answer is positive

1. $\int x \ln(x)^3 \, dx$ integration by parts.

$$u = \ln(x)^3 \qquad dv = x\,dx$$
$$du = 3\frac{\ln(x)^2}{x}dx \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln(x)^3 - \frac{3}{2}\int x\ln(x)^2\,dx$$ do it again!

$$u = \ln(x)^2 \qquad dv = x\,dx$$
$$du = 2\frac{\ln(x)}{x}dx \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln(x)^3 - \frac{3}{2}\left[\frac{x^2}{2}\ln(x)^2 - \int x\ln(x)\,dx\right]$$ one last time!

$$u = \ln(x) \qquad dv = x\,dx$$
$$du = \frac{1}{x}dx \qquad v = x^2/2$$

$$= \frac{x^2}{2}\ln(x)^3 - \frac{3}{2}\left[\frac{x^2}{2}\ln(x)^2 - \left[\frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx\right]\right]$$

$$= \frac{x^2}{2}\ln(x)^3 - \frac{3}{4}x^2\ln(x)^2 + \frac{3}{4}x^2\ln(x) - \frac{3}{4}\cdot\frac{x^2}{2} + C$$

$$= \boxed{\frac{3}{4}x^2\left[\frac{2}{3}\ln(x)^3 - \ln(x)^2 + \ln(x) - \frac{1}{2}\right] + C}$$

2. $\int \sin^5 x \, dx = \int \sin x \cdot \sin^2 x \cdot \sin^2 x \, dx$

odd power of sin(x).

use $\sin^2(x) = 1 - \cos^2 x$

$$= \int \sin x (1-\cos^2 x)(1-\cos^2 x) \, dx$$

let $u = \cos x$

$du = -\sin x \, dx$

$dx = \frac{-du}{\sin x}$

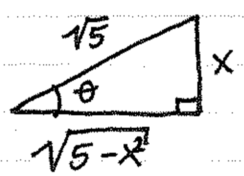
$$= -\int (1-u^2)(1-u^2) \, du$$

$$= -\int 1 - 2u^2 + u^4 \, du = -u + \frac{2}{3}u^3 + \frac{u^5}{5} + c$$

$$= \boxed{-\cos x + \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$$

3. $\int \frac{dx}{x^2\sqrt{5-x^2}}$ trig substitution.

draw the triangle



$\sin\theta = \frac{x}{\sqrt{5}}$

my substitution.

$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta \, d\theta$

$\cos\theta = \frac{\sqrt{5-x^2}}{\sqrt{5}}$ $\quad \frac{1}{\sqrt{5-x^2}} = \frac{1}{\sqrt{5}\cos\theta}$

$$\int \frac{1}{x^2} \cdot \frac{1}{\sqrt{5}\cos\theta} \sqrt{5}\cos\theta \, d\theta = \int \frac{1}{x^2} \, d\theta$$

$$= \int \frac{1}{5\sin^2\theta} \, d\theta = \frac{1}{5}\int \csc^2\theta \, d\theta$$

look this one up!

$$= -\frac{1}{5}\cot\theta + c = -\frac{1}{5}\frac{\cos\theta}{\sin\theta} + c$$

$$= -\frac{1}{5}\frac{\sqrt{5-x^2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{x} + c = \boxed{-\frac{1}{5}\frac{\sqrt{5-x^2}}{x} + c}$$

4. $\displaystyle\int \frac{x}{\sqrt{5-x^2}}\,dx$ u-substitution.

let $u = 5-x^2$
$du = -2x\,dx$
$dx = \dfrac{-du}{2x}$

$$\int \frac{x}{\sqrt{u}} \frac{-du}{2x} = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\left[2u^{1/2}\right] + C = \boxed{-\sqrt{5-x^2} + C}$$

5. $\displaystyle\int \frac{dx}{x(x-1)^2}$ do partial fractions to simplify.

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$1 = A(x-1)^2 + Bx(x-1) + Cx$$
$$1 = A(x^2-2x+1) + B(x^2-x) + Cx$$
$$1 = (A+B)x^2 + (C-2A-B)x + A$$

$A+B=0$ $\quad A=-B \quad \boxed{B=-1}$

$C-2A-B=0$ $\quad C-2+1=0$

$\boxed{A=1}$ $\quad \boxed{C=1}$

$$\int \frac{dx}{x(x-1)^2} = \int \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \boxed{\ln|x| - \ln|x-1| + \left[\frac{-1}{(x-1)}\right] + C}$$

6. $\displaystyle\int \sin x \cos^3 x \, dx$ u-substitution

let $u = \cos x$
$du = -\sin x \, dx$

$$= -\int u^3 \, du = -\frac{u^4}{4} + c = \boxed{-\frac{1}{4}\cos^4 x + c}$$

7. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ u-substitution

let $u = 4-x$
$du = -dx$

remember to change limits!

$x = 0 \longrightarrow u = 4$
$x = 4 \longrightarrow u = 0$

$$-\int_4^0 u^{-1/2} \, du$$

$$= \int_0^4 u^{-1/2} \, du = 2u^{1/2} \Big|_0^4 = 2\sqrt{4} - 0 = \boxed{4}$$

8. $\displaystyle\int_1^\infty x^{-19/20} \, dx$ improper.

$$\int_1^b x^{-19/20} \, dx = 20\, x^{1/20} \Big|_1^b = 20\, b^{1/20} - 20$$

$$\lim_{b \to \infty} 20\, b^{1/20} - 20 = \boxed{\text{DIVERGES}}$$