

Math for Data Science

Calculus - Limits

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Important Information

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- Office Hours take place in Duke 209 unless otherwise noted –
Office Hours Schedule

Today's Goals:

- Start our discussion of topics in Calculus.
- Taking Limits

Where are we?

So far we have reviewed important concepts to get more comfortable with ideas like:

- Fractions
- Exponents and Logarithms
- Functions and graphing

Where are we?

We applied these ideas to real data using regression:

- We can do feature engineering - apply functions to our data.
- Given data we can fit a curve to it and use it for predictions.

But what more can we do with these curves, functions, ideas?

Calculus is the study of continuous change.

In Data Science we are often interested in how variables change in relation to one another. We know how to find relationships like:

$$y = \beta_0 + \beta_1 f_1(x) + \beta_2 f_2(x) + \dots$$

but what do we do with this information after we have it?

Calculus is the study of continuous change.

- Rates of change or curvature.
- Long term predicted behavior.
- Finding maximums and minimums.
- Developing tools to be used in probability, regression, and machine learning.

Limits

Limits are a mathematical idea that help us understand how a function behaves as we get close to a given point. Limits form the basis for all of calculus! Let's start with some notation:

$$\lim_{x \rightarrow a} f(x)$$

This is just a fancy way to write: “What happens to $f(x)$ as we get really really really close to $x = a$.”

Limits

How would you put the following notation into words?

$$\lim_{x \rightarrow \infty} e^{-x}$$

Limits

$$\lim_{x \rightarrow \infty} e^{-x}$$

To me this reads “What happens to the function e^{-x} as we get really really really close to $x = \infty$ knowing that we can’t ever actually reach infinity.

Limits

It is a persnickety point, but when we think about limits we don't ever actually let x reach the point. We are just thinking about what happens as we get super close to the point from either side.

Limits

To find a limit we imagine getting closer to the given point a from both the left and right side.

- 1 The answer is just a number N , then we say

$$\lim_{x \rightarrow a} f(x) = N$$

- 2 The answer just gets more and more positive or more and more negative, never stopping, then we say

$$\lim_{x \rightarrow a} f(x) = DNE$$

- 3 In some cases the answer fluctuates, but we can't pinpoint a single number, then we say

$$\lim_{x \rightarrow a} f(x) = DNE$$

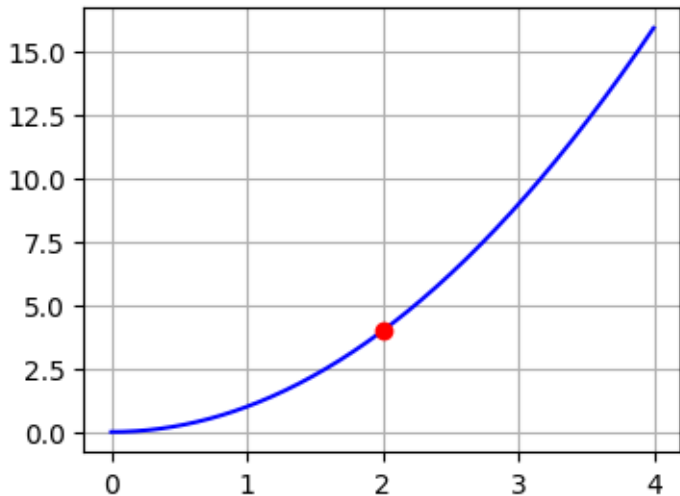
An Example

$$\lim_{x \rightarrow 2} x^2$$

You can probably guess what the result is, but we are going to explore this in excruciating detail just for fun!!!

- 1 Look at a graph and imagine walking closer and closer to $x = 2$.
- 2 See what happens as we plug x values closer and closer to $x = 2$ into our function $f(x) = x^2$. Notice we could approach $x = 2$ from the right or the left!

An Example



An Example - From the left

$x=0$ and $f(x) = 0$

$x=1$ and $f(x) = 1$

$x=1.5$ and $f(x) = 2.25$

$x=1.75$ and $f(x) = 3.0625$

$x=1.875$ and $f(x) = 3.515625$

$x=1.9375$ and $f(x) = 3.75390625$

$x=1.96875$ and $f(x) = 3.8759765625$

$x=1.984375$ and $f(x) = 3.937744140625$

$x=1.9921875$ and $f(x) = 3.96881103515625$

$x=1.99609375$ and $f(x) = 3.9843902587890625$

Continue the limit (yes/no): no

An Example - from the right

$x=4$ and $f(x) = 16$

$x=3$ and $f(x) = 9$

$x=2.5$ and $f(x) = 6.25$

$x=2.25$ and $f(x) = 5.0625$

$x=2.125$ and $f(x) = 4.515625$

$x=2.0625$ and $f(x) = 4.25390625$

$x=2.03125$ and $f(x) = 4.1259765625$

$x=2.015625$ and $f(x) = 4.062744140625$

$x=2.0078125$ and $f(x) = 4.03131103515625$

$x=2.00390625$ and $f(x) = 4.0156402587890625$

Continue the limit (yes/no): no

An Example

No matter which way we look at it we see

$$\lim_{x \rightarrow 2} x^2 = 4$$

Now I totally remember thinking: **WHY NOT JUST PLUG IN 4!?!?!?! Well, this is because there are cases where this would get your in trouble and it is not the mathematically correct interpretation of the limit.**

You Try

Redo the analysis above for the following limits:

$$\lim_{x \rightarrow 2} x^3$$

$$\lim_{x \rightarrow 1} \ln(x)$$

$$\lim_{x \rightarrow 0} \frac{1}{x}$$

What happens in this last case? Do you get an error in the code?

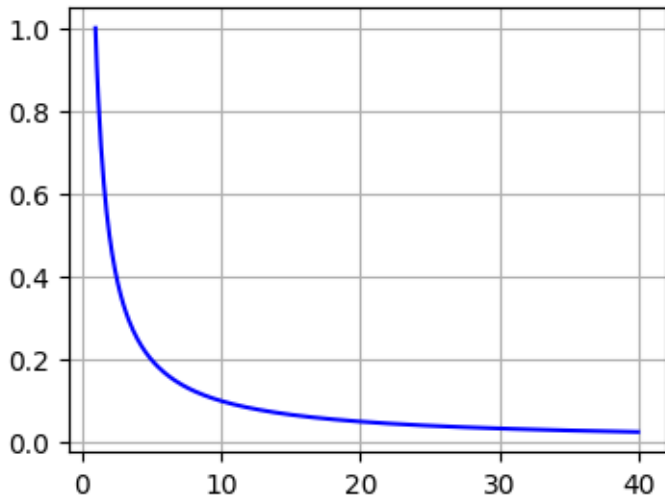
Limits at Infinity

Many times we want to know what happens as x gets really big or really small:

$$\lim_{x \rightarrow \infty} \frac{1}{x}$$

in these cases we let x get bigger and bigger, and just approach from one side. Lets explore this one

Limits at Infinity



Limits at Infinity - from the left

$x=100$ and $f(x) = 0.01$

$x=101$ and $f(x) = 0.009900990099009901$

$x=103$ and $f(x) = 0.009708737864077669$

$x=107$ and $f(x) = 0.009345794392523364$

$x=115$ and $f(x) = 0.008695652173913044$

$x=131$ and $f(x) = 0.007633587786259542$

$x=163$ and $f(x) = 0.006134969325153374$

$x=227$ and $f(x) = 0.004405286343612335$

$x=355$ and $f(x) = 0.0028169014084507044$

$x=611$ and $f(x) = 0.0016366612111292963$

Continue the limit (yes/no): no

Limits at Infinity

Here we see that as $x \rightarrow \infty$ our function $\frac{1}{x}$ gets smaller and smaller toward zero. So we would say:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

You Try

Redo the analysis above for the following limits:

$$\lim_{x \rightarrow \infty} e^{-x}$$

$$\lim_{x \rightarrow -\infty} x^3$$

$$\lim_{x \rightarrow \infty} \sin(x)$$

Remember if you can't get just one single number, then the limit does not exist.

More complicated Limits

We can take limits of more complicated combinations of functions. To do this we need algebra, some properties of limits, and some instincts about functions.

More complicated Limits - Algebra

Sometimes we can just use algebra to simplify our limits:

$$\lim_{x \rightarrow \infty} \frac{x}{x^2}$$

well this is the same as

$$\lim_{x \rightarrow \infty} \frac{1}{x}$$

which we solved above!

More complicated Limits - Properties of limits

Assume that

$$\lim_{x \rightarrow a} f(x) = L \qquad \lim_{x \rightarrow L} g(x) = M$$

1 Sum and Difference Property

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M.$$

2 Product Property:

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M.$$

More complicated Limits - Properties of limits

- 3 **Quotient Property:** (provided that $M \neq 0$)

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}.$$

- 4 **Constant Multiple Property:**

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x) = c \cdot L,$$

where c is a constant.

More complicated Limits - Properties of limits

Find the limit:

$$\lim_{x \rightarrow \infty} \frac{x+1}{x^2}$$

use algebra to break it into pieces and then use the sum property:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x+1}{x^2} &= \lim_{x \rightarrow \infty} \left(\frac{x}{x^2} + \frac{1}{x^2} \right) = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{1}{x^2} \right) \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0 + 0 = 0\end{aligned}$$

You Try

Find the limit:

$$\lim_{x \rightarrow 3} \frac{x^2 + 5x}{x}$$

More complicated Limits - Instincts about functions.

There are some really interesting special cases:

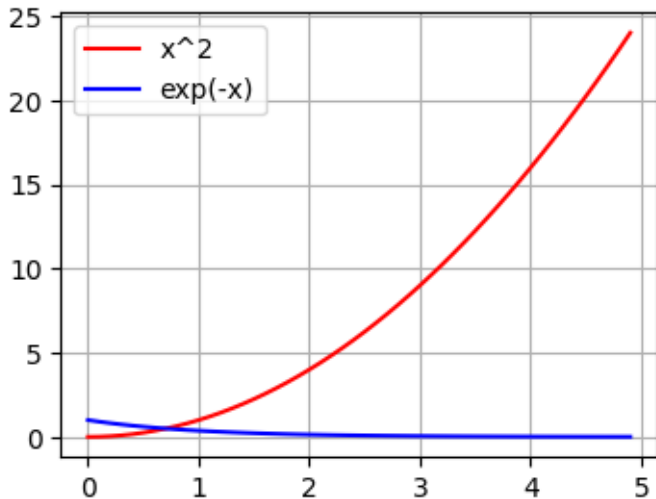
$$\lim_{x \rightarrow \infty} x^2 e^{-x}$$

well if we use our product property above then we see

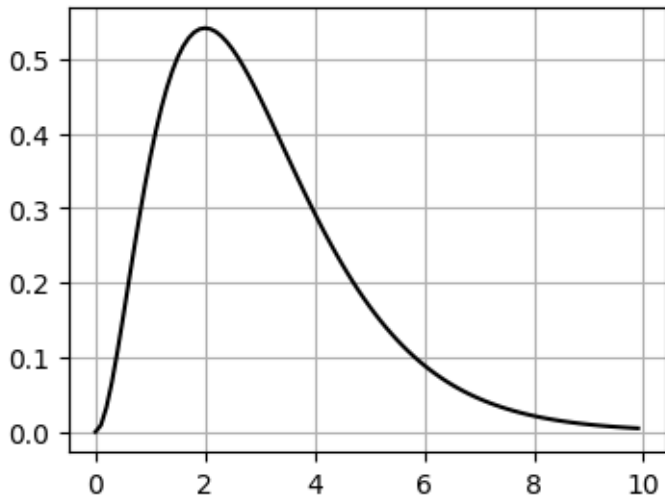
$$\left(\lim_{x \rightarrow \infty} x^2 \right) \left(\lim_{x \rightarrow \infty} e^{-x} \right) = (\infty)(0) = ?$$

for limits like these we have to take into consideration “how fast” or “how dominant” a function is. Lets look at $y = x^2$ and $y = e^{-x}$ using a graph and ask the question: Does x^2 reach infinity before e^{-x} gets to zero?

More complicated Limits - Instincts about functions.



More complicated Limits - Instincts about functions.



More complicated Limits - Instincts about functions.

In the graphs above you can see that the decaying exponent function is more powerful! It drags that limit down to zero before x^2 can reach infinity. If we look at the two together, we see that this is what happens. At first the function increases thanks to the x^2 but then it quickly goes to zero. So in this case:

$$\left(\lim_{x \rightarrow \infty} x^2\right) \left(\lim_{x \rightarrow \infty} e^{-x}\right) = (\infty)(0) = 0$$

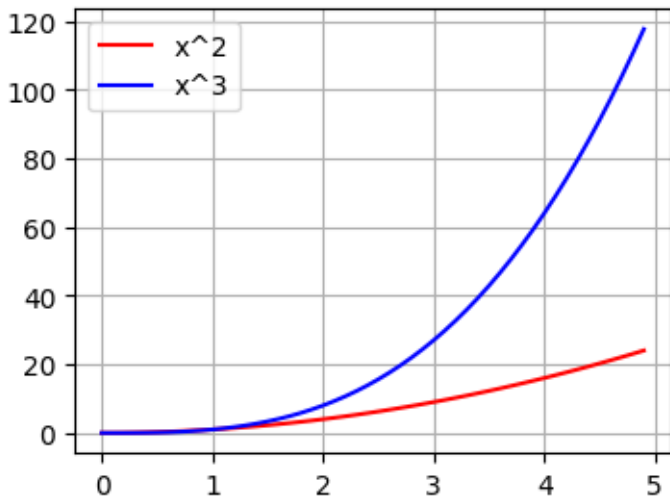
More complicated Limits - Instincts about functions.

we can use this kind of instinct to manage problems like:

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 10}{x^3 + 5x} = \frac{\infty}{\infty} = ?$$

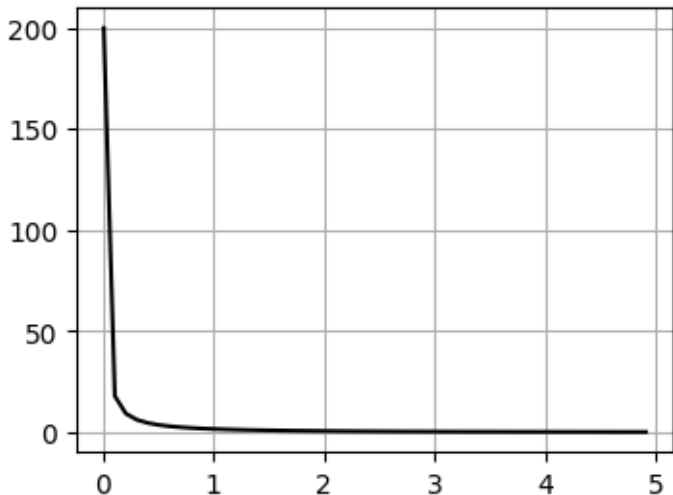
if I think about how quickly the leading functions grow or decay then I can suss out the limit.

More complicated Limits - Instincts about functions.



Since the x^3 gets bigger faster, I think it will reach infinity first so this

More complicated Limits - Instincts about functions.



More complicated Limits - Sympy

Sympy has a function `sp.limit()` that will calculate your limits for you.

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 10}{x^3 + 5x}$$

```
# Define the variable
x = sp.symbols('x')
# Define the function:
f = (x**2-x+10)/(x**3+5*x)

# Calculate the limit of f as x approaches a
# limit(f,x,a)
sp.limit(f, x, np.inf)
```

0

so this means

More complicated Limits - Sympy

What about this one:

$$\lim_{x \rightarrow \infty} \sin(x)$$

What does the output here mean?

$$\langle -1, 1 \rangle$$

Conclusion

Limits are a way to explore functions and think about what they do as they reach certain points or even go to infinity. What you should be able to do:

- 1 Figure out the limits of our basic functions using either a graph, your instincts, or by plugging in numbers closer and closer to $x = a$
- 2 Use algebra, properties of limits, and function instincts to calculate more complicated limits.
- 3 Check your results using Sympy.

You Try

Find the limits. In each case graph the function and talk about what the limit means in that case (aka... I am getting closer and closer to ...). Then figure out the limit using algebra and the properties of the limit. Finally check your answer using Sympy.

1

$$\lim_{x \rightarrow 2} (x^2 - x + 10)$$

2

$$\lim_{x \rightarrow 0} \frac{x}{x^3 + 1}$$

3

$$\lim_{x \rightarrow \infty} e^{-x/2}$$

4

$$\lim_{x \rightarrow -\infty} \cos(x)$$