Math for Data Science

Calculus - Optimization

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Important Information

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- Office Hours take place in Duke 209 unless otherwise noted – Office Hours Schedule

Today's Goals:

- Use our understanding of derivatives for optimization
- Review - First derivative test
- Second Derivative
- Optimization example

Derivative Definition

$$\frac{dy}{dx} = \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx}$$

First Derivative Test

- The function has a **local maximum or minimum whenever the derivative equals zero**:

$$\frac{dy}{dx} = 0$$

we solve this for all of the x-values where this is true, x^*.

- The points $(x^*, y(x^*))$ where the derivative equations zero are called “stationary points” or “critical points”
- We can tell if a critical point is a **local max** by the **derivative going from increasing to decreasing**, positive to negative, as we cross the point.
- We can tell if a critical point is a **local min** by the **derivative going from decreasing to increasing**, negative to positive, as we cross the point.

Global Maximum and Minimum

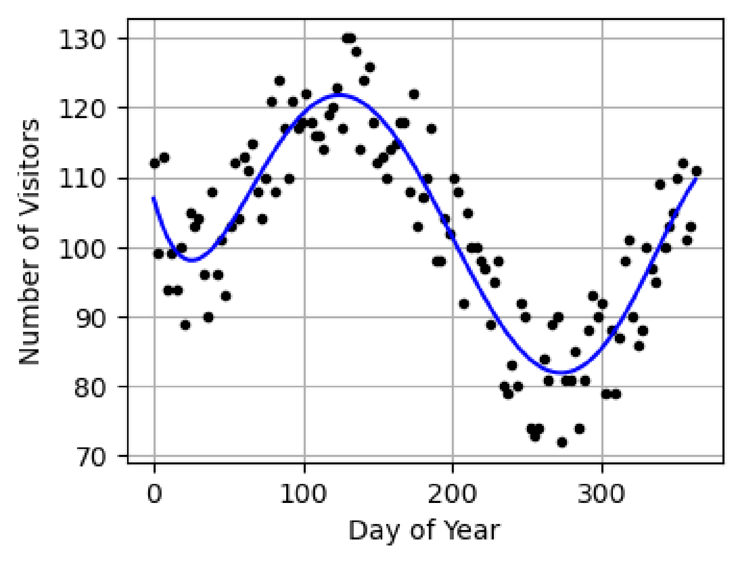
Check y-values of all of the critical points but **also check the end points** or take the limit if there are no clear endpoints.

Can we take the derivative of the derivative?

Questions

- What does the derivative tell us?
- What does it mean to take the derivative of the derivative?
- What is the rate of change of the rate of change?

Look at the graph of $y = f(x)$



The second derivative

If the derivative tells us about the change in the function or the slope...
what does the change in the derivative tell us?

Video:

https://youtu.be/QeKJWYxAvBM

The second derivative tells us the change in slope

What does a changing slope mean?

- If the slope does not change - then we have a straight line
- If the slope changes by getting more positive - then we have a function that is curving upwards
- If the slope changes by getting more negative - then we have a function that is curving downward

The second derivative tells us about the curvature of the function

The second derivative

The notation for the second derivative:

$$\frac{d^2y}{dx^2}$$

This tells us how curved our function is at different points.

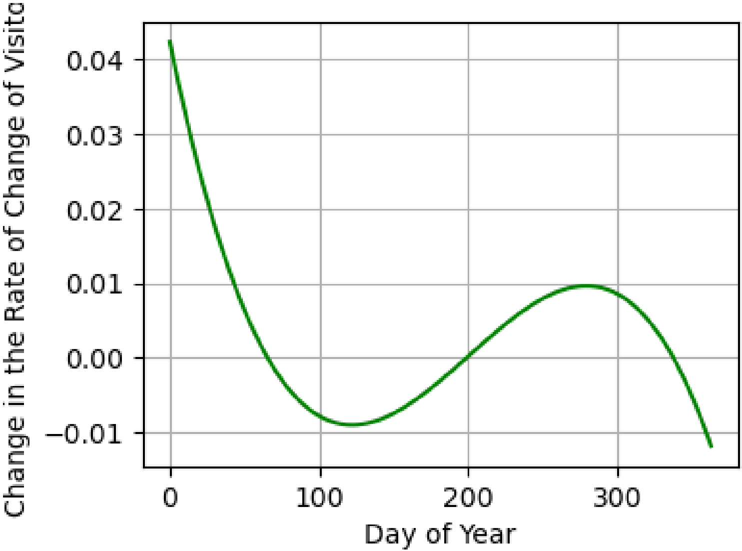
In terms of this application "how quickly is my decrease in visitors changing to an increase in visitors" OR "How fast are we going from increasing to decreasing"

SYMPY can do the second derivative

```
y_pp = sp.diff(y,x,x)
print(y_pp)
```

```
-9.61705202389376e-9*x**3 + 5.79365836899572e-6*x**2 - 0.00098
```

Plot the second derivative

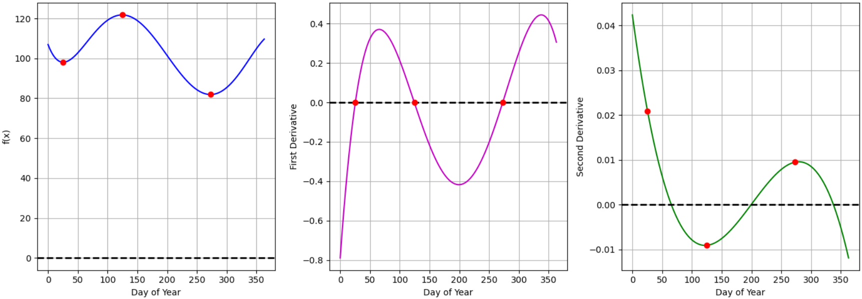


Plot the second derivative

What does this tell me?

- Where is our function the most curvy?
- Where is our function curved upward?
- Where is our function curved downward?
- Where does our function act kinda like a straight line - zero curvature?

Do our answers here make sense?



Can the second derivative say anything about Critical Points?

- If my critical point x^* is a minimum then my second derivative is _______________. (and vice versa)
- If my critical point x^* is a maximum then my second derivative is _______________. (and vice versa)

This is called the second derivative test

You can use either the first or second derivative test to figure out whether a critical point that you found is a maximum or minimum.

Optimization

Optimization means finding maximums and minimums of some function. There are lots of situations where you might want to maximize or minimize something:

Optimization

- **Economics (Thanks Mia!)** - we often want to minimize the cost of production - but this cost depends on the price of various components in the development of the product, the cost of labor (number of employees), and possible the capital cost (debt or rent).

Optimization

- **Medicine** - we often have some idea of how a drug is absorbed or eliminated and we want to make sure the maximum or minimum dose remains below/above some safe/effective amount.

Optimization

- **Video Games** - so many video games are based on optimization - League of Legends: Effective health when defending against physics damage is a function of Armor and Health - which can be purchased. How can you best spend your money?

Economics Example

Say you own a company and and you have been tracking the cost to manufacture your product. The product is made up of the following items

- 4 propellers that cost 2 dollars each
- 1 drone body that costs 4 dollars
- 2 HD cameras that cost 15 dollars each

$46x$

Economics Example

- When we buy more product we can start to negotiate a better price:

$-0.6x^2$

- To keep our office open it costs us 1000 per month

1000

- If we buy too much material it actually starts to cost us money because of storage for the materials and inefficiency in organizing that much stuff:

$0.01x^3$

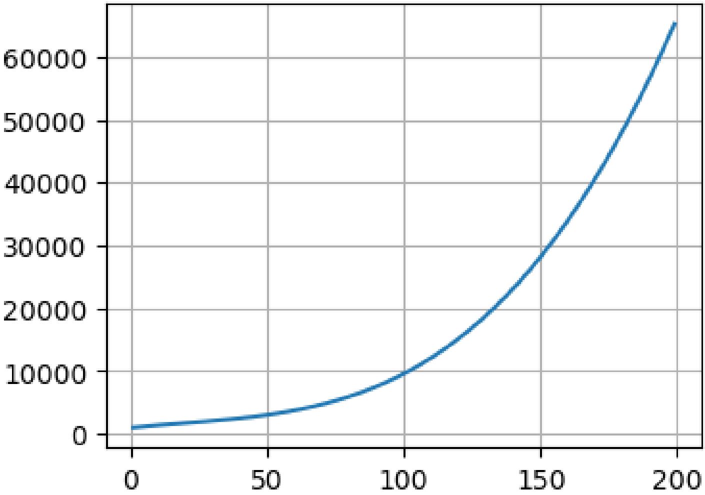
Economics Example

When we add all these ideas together we get a TOTAL cost function:

$$C(x) = 0.01x^3 - 0.6x^2 + 46x + 1000$$

How can we minimize the cost per drone, so that we are getting the biggest return on our work?

Plot the Cost Function



Economics Example

We need to know the cost of one of our drones to figure out the profit!

$$Profit = Revenue - Cost$$

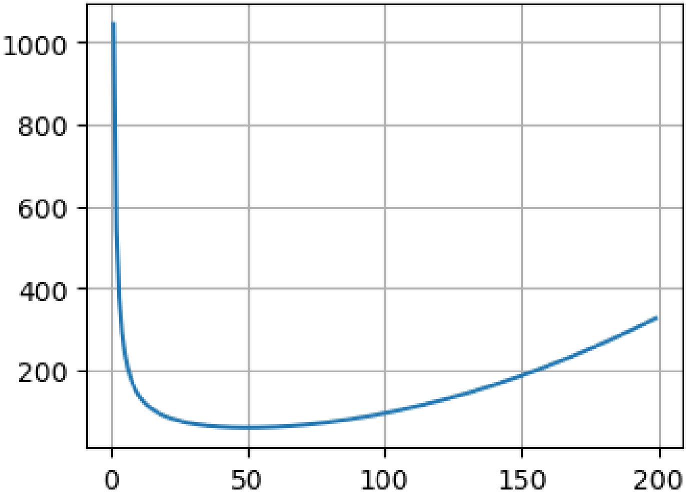
So we want to minimize the cost per drone!

Find the Average Cost per Drone

If x represents number of drones then dividing by x will give us cost per drone.

$$A(x) = \frac{C(x)}{x} = 0.01x^2 - 0.6x + 46 + \frac{1000}{x}$$

Plot the Average Cost per drone



Let's use Derivatives to Minimize!

1. Find the critical point(s)
2. Decide if the represent local max or min
3. Decide if they represent global max or min
4. Interpret the results

See the calculations in the Day 13 Student Notebook!

What do all these calculations tell us?

We only have one real critical point at $x = 50$

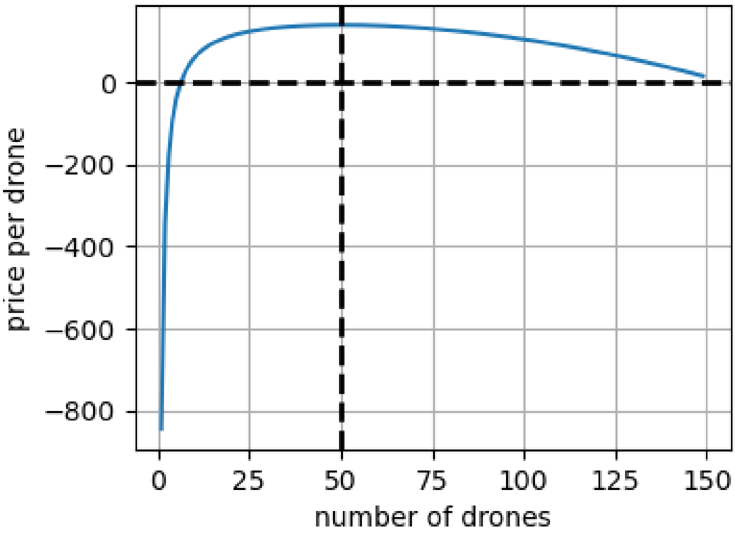
This critical point is a minimum: - First Derivative Test: The slope goes from negative to positive so we are in a valley - Second Derivative Test: The function is curved up so our point is in a valley.

We will have minimum cost per drone if we make 50 of them.

The cost per drone is 61 dollars.

So our profit per drone at this point is $P = 200 - 61 = 139$. We make 139 dollars per drone. Or a total profit per month of $139 * 50 = 6950$.

Here is a plot of the profit per drone



Now it's your turn!

Here are two optimization problems that you can try. Remember our goal is to get a nice function that we can take the derivative of to look for maximum or minimum points. Once we have that we can answer questions!

You try - Drug Absorption Example

Idea from:
https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter

You try - Drug Absorption Example

The concentration of caffeine C in the body at time t is modeled by the function:

$$c(t) = \frac{D}{1 - \frac{\beta}{\alpha}} \left(e^{-\beta t} - e^{-\alpha t}\right)$$

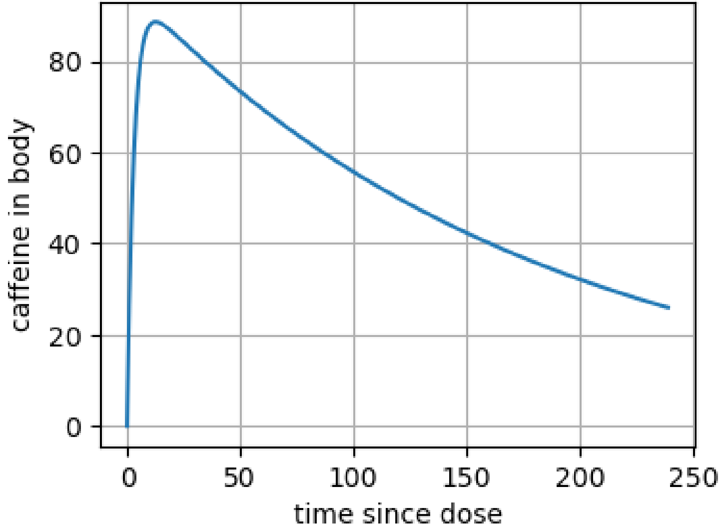
here α is the absorption rate and β is the elimination rate. D is the dose measured in mg. Parameters vary per person but entered below are the ones we will use:

$$\alpha = 0.33$$

$$\beta = 0.33/60$$

$$D = 95$$

Graph the function



Questions:

1. Assuming you drink one cup of coffee at 9am in the morning. When will you have the maximum amount of caffeine in your system? How much is in your system at that time?
2. Does changing the dose change the time at which you will reach your maximum concentration? Redo the analysis for D=200, D=500, and D=0. Explain the results.
3. Now that you know the time at which this function reaches a maximum (for the given α and β values), calculate a minimum safe dose of caffeine. Let's say you have a patient who can only have 25mg of caffeine in their system at one time, how big can their dose be? (HINT, Solve for D by hand, plug in your t^* and let $c(t^*) = 25$)

Make sure to graph the function, the derivative, and the second derivative.

You Try - Video Game Example

Idea from:
https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter

You Try - Video Game Example

In the video game League of Legends, a player's Effective Health when defending against physical damage is given by

$$E = \frac{H(100 + A)}{100}$$

where H is health and A is armor. Health costs 2.5 gold per unit, and Armor costs 18 gold per unit. This is a bit more tricky because we have a function of two variables here. To find a single maximum we need another constraint. Usually in games you are constrained by the amount of gold you have. This is what we will do.

Questions

1. Imagine that you have 3600 gold, and you need to optimize the effectiveness E of your health and armor to survive as long as possible against the enemy team's attacks. How much of each should you buy?
    - First, consider that you can spend your money as

      $$3600 = 2.5H + 18A$$

      since you have 3600 gold and armor and health cost 2.5 and 18 respectively.
    - Solve this for A
    - Plug this into the E equation to get $E = E(H)$
    - Then use optimization to see if there is a maximum health H
    - Once you have that value you can plug back into the A equation to get the amount of armor.

Questions

2. How many Armor points should you buy? How many health points?
3. What is the maximal E that you can afford?
4. How much gold should you spend on armor? How much gold on health?

Make sure to graph the function, the derivative, and the second derivative.