Math for Data Science

Eigenvalues and Determinants

Joanna Bieri DATA100

Important Information

- Email: joanna_bieri@redlands.edu
- Office Hours take place in Duke 209 unless otherwise noted – Office Hours Schedule

Today's Goals:

- Determinants
- Eigenvalues
- Applications

Vectors and Matrices (review)

Span of a set of vectors is the whole set of possible vectors that can be created by taking a linear combination of the vectors in the set.

Two vectors are **Linearly Independent** if they do not lie along the same line.

We can tell if two vectors are **Linearly Dependent** because one will be a scalar multiple of the other... they have different magnitudes but the same direction.

Vectors and Matrices (review)

Matrix Dot Product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Vectors and Matrices (review)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Vectors and Matrices (review)

Using Numpy

```
v = np.array([3,2])
A = np.array([[0,-1],[1,0]])
v_new = np.matmul(A,v)
print(v,v_new)
```

Vectors and Matrices (review)

Matrix Identity and Inverse

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = I$$

```
np.linalg.inv(A)
```

Matrix Determinants

Remember from last time that we can use matrix multiplication to transform a vector. When we preform these **linear transformations** we often expand or squish our vector space. The amount of expansion or squishing can be helpful to know.

The **determinant** measures something about the size of a transformation. It measures how much a vector space changes when you apply a transformation. Are we squishing the vectors, expanding them, shearing them?

These also can be calculated by hand but we will use numpy in this class.

Matrix Determinants

In two dimensions the determinant is give by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a * d - b * c$$

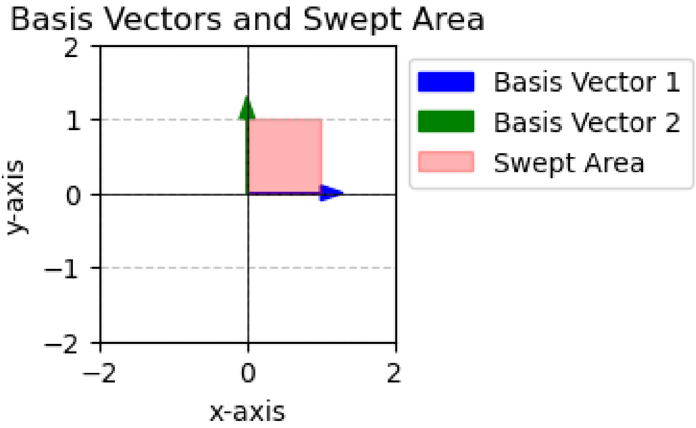
We will consider the determinants of a few different matrices and see how they correspond to the transformations they represent.

Example 1

We will start by considering our basis vector and look at the are swept out.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Determinants



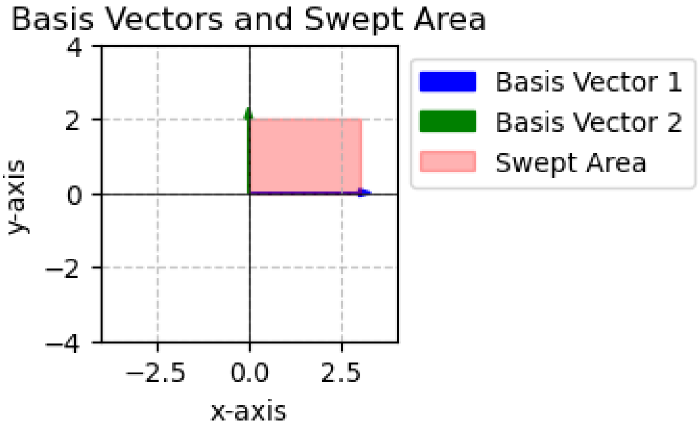

```
The area in red is 1.0 square units
```

Matrix Determinants

Here we see that the area in red is 1 square unit. What happens if we apply a matrix multiplication?

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Matrix Determinants




```
The area in red is 6.0 square units
```

Matrix Determinants

Notice how the area changed. We expanded the length of both of our vectors this also expanded the space. How much was the space expanded?

Matrix Determinants

Determinant

Consider the determinant of the matrix that defined our transformation

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = 3 * 2 - 0 * 0 = 6$$

Notice that this is the exact amount that we expanded our vector space!

Matrix Determinants

Example 2

Should a simple rotation change the area? Let's see

Consider the rotation matrix from last time

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Matrix Determinants

Calculate the determinant by hand and compare to the numpy code below

```
A = np.array([[0,-1],[1,0]])
DET = np.linalg.det(A)
DET
```

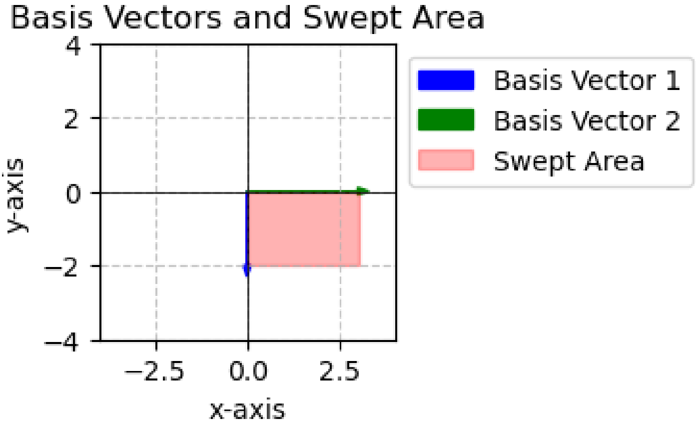
```
1.0
```

Matrix Determinants

This tells me that the overall area should not change if we apply this transformation. Let's see this in practice.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}$$

Matrix Determinants




```
The area in red is 6.0 square units
```

The area did not change, but it was rotated to a new location!

You Try:

Starting with the following matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For each matrix multiplication below:

- Find the determinant of the transformation given
- Say in words what will happen to the area
- Do the matrix multiplication to the the new matrix
- Use the code to plot the new matrix and area.

You Try:

1. $$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

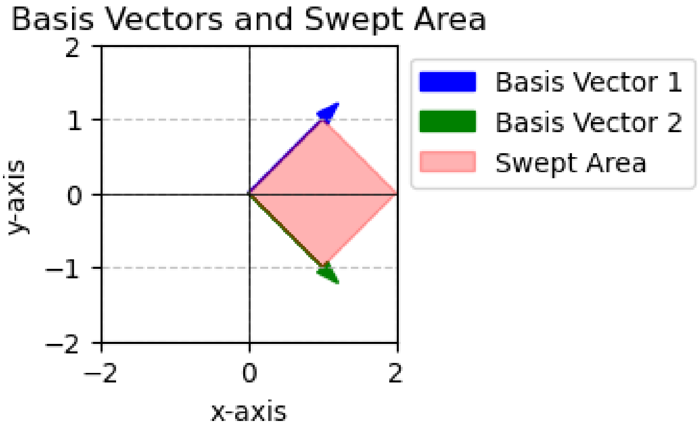
2. $$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

You Try:

3. $$\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

You Try:




```
The area in red is 2.0 square units
```

Determinant of Linearly Dependent vectors

Consider the vectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

First, how can you tell by looking that these vectors are linearly dependent?

Determinant of Linearly Dependent vectors

Lets try to take the determinant of these two vectors:

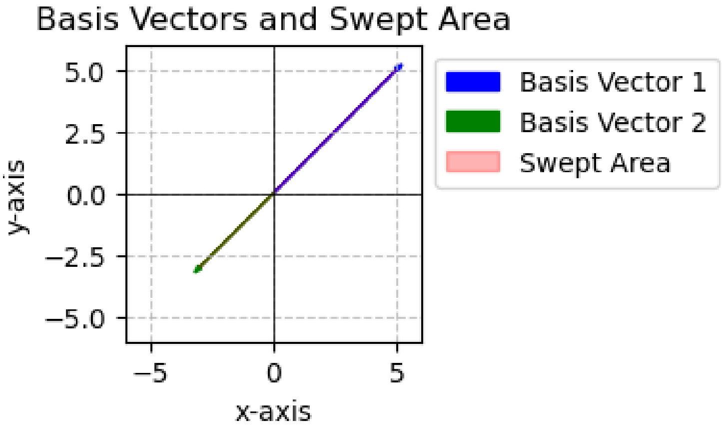
$$\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} = 1 * 4 - 1 * 4 = 0$$

What does this mean?

Determinant of Linearly Dependent vectors

$$\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 5 & -3 \end{bmatrix}$$

Determinant of Linearly Dependent vectors




```
The area in red is 2.2204460492503135e-15 square units
```

Determinant of Linearly Dependent vectors

Whenever two vectors are linearly dependent the determinant is zero! If we try to apply a transformation that has a zero determinant, the we are reducing our space to a single line. The area between the vectors is zero because they are on the same line!

Special Kinds of Matrices

Square

Equal number of rows and columns

$$\begin{bmatrix} 4 & 2 & 4 \\ 5 & 3 & 7 \\ 9 & 3 & 6 \end{bmatrix}$$

Special Kinds of Matrices

Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special Kinds of Matrices

Inverse

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 5 & 3 & 7 \\ 9 & 3 & 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{1}{3} \\ \frac{11}{2} & -2 & \frac{-4}{3} \\ -2 & 1 & \frac{1}{3} \end{bmatrix}$$

Special Kinds of Matrices

Diagonal

A diagonal matrix has values only along the diagonal and zeros everywhere else. Diagonal matrices are desirable in many computational settings because simplify the calculations.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Special Kinds of Matrices

Triangular

Triangular matrices have non-zero values along the diagonal and non-zero values either above or below the diagonal with only zeros on the other side. They represent easy to solve systems and so are nice to work with.

Special Kinds of Matrices

Upper Triangular

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

Special Kinds of Matrices

Lower Triangular

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 0 \\ 9 & 3 & 6 \end{bmatrix}$$

Special Kinds of Matrices

Sparse Matrix

A sparse matrix is one that contains mostly zeros. From a computational standpoint the are very efficient since we don't have to waste memory storing values that we know are zero.

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors are an important idea in applied linear algebra! The allow us to find the growth or decay rates in linear systems, construct solutions to systems, and to decompose matrices into basic components.

Eigenvalues and Eigenvectors

Eigenvalues and vectors are defined by the following equation

$$A\vec{v} = \lambda\vec{v}$$

Where λ is an eigenvalue and $\vec{v}$ is an eigenvector. Notice what this equation is saying: Given a matrix A we are looking for a pair $\vec{v}$ and λ such that when we multiply $A\vec{v}$ we get back just a scalar multiple of $\vec{v}$.

Eigenvalues and Eigenvectors

Let's see this in action! Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

Eigenvalues and Eigenvectors

we will use numpy to find eigenvalues

```
A = np.array([[1,2],[4,5]])
eigenvalues, eigenvectors = np.linalg.eig(A)

print("EIGENVALUES")
print(eigenvalues)

print("EIGENVECTORS")
print(eigenvectors)
```

```
EIGENVALUES
[-0.46410162  6.46410162]
EIGENVECTORS
[[-0.80689822 -0.34372377]
 [ 0.59069049 -0.9390708 ]]
```

Eigenvalues and Eigenvectors

Here we get two eigenvalues and two eigenvectors. **THEY GO TOGETHER** so we have

$$\lambda_1 = -0.4641016 \quad \vec{v}_1 = \begin{bmatrix} -0.80689822 \\ 0.59069049 \end{bmatrix}$$

$$\lambda_2 = 6.46410162 \quad \vec{v}_1 = \begin{bmatrix} -0.34372377 \\ -0.9390708 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Lets see what happens when we apply these to the equation that defines our eigenvalues and vectors

```
eigv =np.array([-0.80689822,0.59069049])
A = np.array([[1,2],[4,5]])

np.matmul(A,eigv)
```

```
array([ 0.37448276, -0.27414043])
```

```
lamb1 = -0.46410162
eigv =np.array([-0.80689822,0.59069049])
lamb1*eigv
```

```
array([ 0.37448277, -0.27414041])
```

Matrix Decomposition

Now what we can find eigenvalues and eigenvectors we can actually deconstruct A in an interesting way:

$$A = Q\Lambda Q^{-1}$$

where Q contains the eigenvectors and Λ has the eigenvalues on the diagonal. This is like factoring a matrix to make it easier to deal with.

Matrix Decomposition

Let's look at these objects

```
# Construct Q
Q = eigenvectors
Q
```

```
array([[-0.80689822, -0.34372377],
       [ 0.59069049, -0.9390708 ]])
```

Matrix Decomposition

```
# Construct Lambda Matrix
Lambda = np.array([[eigenvalues[0],0],[0,eigenvalues[1]]])
Lambda
```

```
array([[-0.46410162,  0.        ],
       [ 0.        ,  6.46410162]])
```

Matrix Decomposition

```
# Find the inverse Q^-1
Qinv = np.linalg.inv(Q)
Qinv
```

```
array([[-0.97741588,  0.35775904],
       [-0.61481016, -0.8398463 ]])
```

Matrix Decomposition

And we can check that this actually works!

```
Q@Lambda@Qinv
```

```
array([[1., 2.],
       [4., 5.]])
```

Summary

We now have the power to take determinants which help us test for linear dependence and tell us about the size of the transformation that we are applying to the vector space.

We can also find eigenvalues and eigenvectors and use these things to decompose a matrix system.

What's next Now that we can calculate eigenvalues and eigenvectors we can sue these ideas to reduce the dimensionality of data and to transform our vectors space for better classification algorithms. This is what we will do next time!

You Try

Here are a few different matrix problems for you to practice with:

1. Are the following two vectors linearly dependent? Use the determinant test:

$$v1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad v2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

2. Are the following two vectors linearly dependent? Use the determinant test:

$$v1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

You Try

3. Consider the transformation given by the matrix

$$v1 = \begin{bmatrix} 1, 2 \\ 2, 6 \end{bmatrix}$$

find the determinant and talk about the change in the area. Apply this transformation to the basis:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and plot your results.

You Try

4. Use linear algebra to solve the system of equations for x,y, and z:

$$3x + 1y + 0z = 54$$

$$2x + 4y + 1z = 12$$

$$3x + 1y + 8z = 6$$

you will first need to write the system in matrix form.

You Try

5. Consider the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of this matrix. Write them as pairs $\lambda_1, \ \vec{v_1}$ and $\lambda_2, \ \vec{v_2}$

6. Using the values you found in 5. write the system in decomposed form

$$A = Q\Lambda Q^{-1}$$

show values for Q, Q^{-1} and Λ.