Solving gives $C_1 = \frac{5+\sqrt{5}}{10}$ and $C_2 = \frac{5-\sqrt{5}}{10}$. Thus the closed-form solution to the Fibonacci recurrence relation is

$$x(n) = \frac{5+\sqrt{5}}{10}\left(\frac{(1+\sqrt{5})}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{(1-\sqrt{5})}{2}\right)^n.$$

One amazing feature of this solution is that despite the fact that all of its factors are irrational, it takes on integer values for every n .

1.4.4.1 A Model for Annual Plants. Loosely speaking, linear models are appropriate when the model parameters are constants. More precisely, a model is linear if the amount of some item is a constant times the amount last year, or the sum of constants times the amount of the item for several past years. One very nice example of this type is a model for the propagation of annual plants. This example is taken from *Mathematical Models in Biology* by Edelstein-Keshet [15].

Consider an annual plant which germinates in the spring, blooms in early summer, and produces seeds in early fall. A fraction of these seeds survive the winter (i.e., do not rot, are not eaten, etc.) and a fraction of these seeds germinate the next spring, flower, and produce more seeds. Of the seeds that do not germinate, a fraction survive the winter, and a fraction of these germinate. This process could continue, but we assume that the number of seeds which survive more than two winters and germinate are negligible.

We make the following parameter and variable assignments:

1. $p(n)$ is the number of plants in year n.
2. γ is the average number of seeds produced per plant during a year.
3. σ is the fraction of the seeds which survive a winter
4. α is the fraction of one-year-old seeds which germinate in the spring
5. β is the fraction of two-year-old seeds which germinate in the spring

Next we set up the equation for the annual plant model. How does a new plant get formed? Either a one-year-old seed germinates or a two-year-old seed germinates. Thus $p(n)$ = plants from one-year-old seeds plus the plants from two-year-old seeds. A plant germinates from a one-year-old seed if last year a plant produced seeds, which survived the winter, and then germinated. Last year there were $p(n-1)$ plants, that produced $\gamma p(n-1)$ seeds, of these $\sigma\gamma p(n-1)$ survived the winter, thus $\alpha\sigma\gamma p(n-1)$ new plants germinated from one-year-old seed. Similarly to be a new plant from a two-year-old seed, a plant two years ago ($p(n-2)$) produces seeds (γ), which survive a winter (σ), do not germinate ($1-\alpha$), survive another winter (σ), and then germinate (β). Thus the number of new plants germinating from two-year-old seed is $\beta\sigma(1-\alpha)\sigma\gamma p(n-2)$. Putting these two parts together yields

$$p(n) = \alpha\sigma\gamma p(n-1) + \beta\sigma(1-\alpha)\sigma\gamma p(n-2).$$

This is, of course, a second order linear homogeneous recurrence equation with constant coefficients. To simplify notation, let $a = \alpha\sigma\gamma$ and $b = \beta\sigma(1-\alpha)\sigma\gamma$. The equation is now $p(n) - ap(n-1) - bp(n-2) = 0$. Its characteristic equation is $\lambda^2 - a\lambda - b = 0$

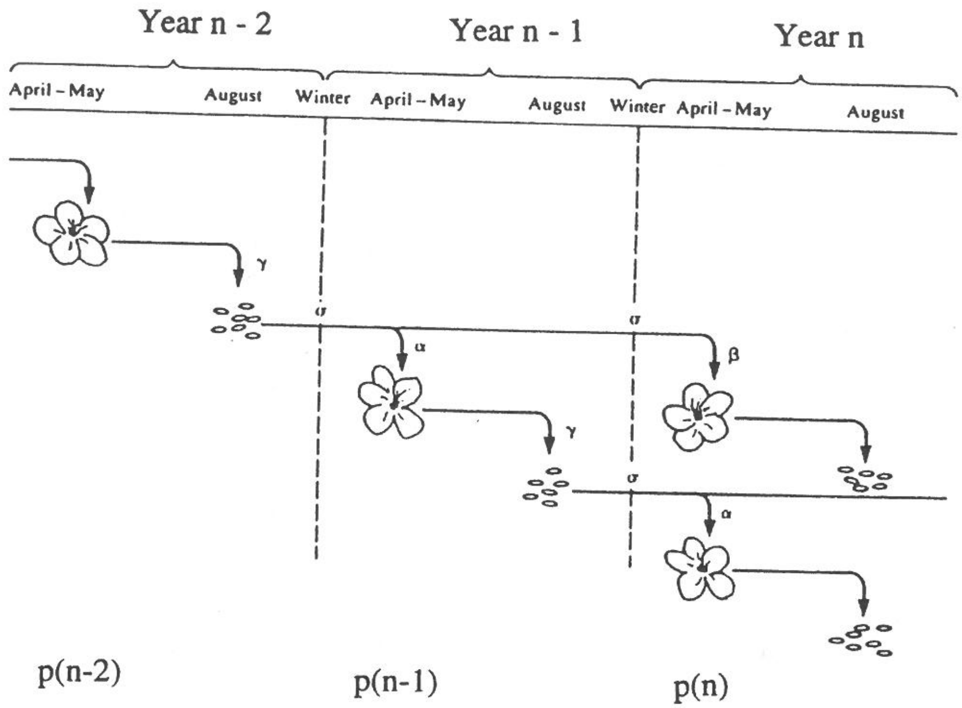

FIGURE 1.17 Annual Plants. Figure reproduced from *Mathematical Models in Biology* by Edelstein-Keshet, 1988, McGraw-Hill, Inc. Used with Permission of The McGraw-Hill Companies.

and its growth rates or eigenvalues are

$$\lambda = \frac{a \pm \sqrt{a^2 + 4b}}{2}.$$

Replacing a and b yields

$$\lambda = \frac{\alpha\sigma\gamma \pm \sqrt{(\alpha\sigma\gamma)^2 + 4\beta\sigma^2(1-\alpha)\gamma}}{2}.$$

With specific values of the parameters, we now find numerical values for the growth rates (eigenvalues) and determine from them whether the plant population is growing or declining, find the constants C_1 and C_2 to complete the closed-form solution, and use either the recursive- or closed-form equation to project the population of the plants.

Suppose, for example, $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 2$, and $\sigma = 0.8$. We compute $\lambda_1 = -0.166$ and $\lambda_2 = 0.966$. The dominant eigenvalue is λ_2, which is less than one, so the population is declining. If all we are interested in is whether the plants will survive, we are finished. They will not. If we are interested in predicting the population size over the next several years, maybe to determine when half of the plants remain, then we need to find either the closed-form solution, or run a simulation. Suppose that currently ($t = 0$) there are 95 plants and that last year there were 100 plants. We want to track the