Lecture 5. Math Modeling.

Last Time: Constant Growth Population Models
Compartmental Diagrams.

Project 2
Due in 1 Week.
10/4

Final project ideas
due 10/6.

TODAY: Closed Form Solutions and
Mathematical Analysis.

Consider our model from last time:

$$P(n+1) = P(n) + rP(n)$$
$$P(0) = P_0$$

Say we want to calculate P(100). That means we need P(99), P(98) ... This is a lot of work!

Sometimes we can "SOLVE" our recorrence relations.
This is called a CLOSED FORM SOLUTION.

<u>EX</u> $P(n+1) = P(n) + rP(n)$

1. Simplify as much as possible

$$P(n+1) = (1+r)P(n)$$

2. Write out the first step

$$P(1) = (1+r)P(0) = (1+r)P_0$$

3. Write out the next few steps.

$$P(2) = (1+r)P(1) = (1+r)(1+r)P_0$$
$$P(3) = (1+r)P(2) = (1+r)(1+r)(1+r)P_0$$

4. Look for a pattern: $P(n) = (1+r)^n P_0$ Exponential Growth model

Now we can calculate any step without needing the previous ones. We can also plot P vs. n on a calculator.

<u>EX</u> $P(n+1) = P(n) + rP(n) + h$ Find a closed form Soln.

1. Simplify: $P(n+1) = (1+r)P(n) + h$
   let $R = 1+r$

$$P(n+1) = RP(n) + h \quad \text{AFFINE eqn.}$$

2. First Step: $P(1) = RP(0) + h = RP_0 + h$

3. Next Steps:

$$P(2) = RP(1) + h = R(RP_0 + h) + h$$
$$= R^2 P_0 + Rh + h$$
$$= R^2 P_0 + (R+1)h$$
$$P(3) = RP(2) + h = R(R^2 P_0 + (R+1)h) + h$$
$$= R^3 P_0 + (R^2 + R)h + h$$
$$= R^3 P_0 + (R^2 + R + 1)h$$

4. Look for the pattern: $P(n) = R^n P_0 + (R^{n-1} + R^{n-2} + \cdots + R^2 + R + 1)h$
   This is our closed form.

This method will work for most first order eqns and very simple higher order eqns.

A SPECIAL CASE: linear recurrence relation with constant coeficients.

$$a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + \cdots + a_{m-1} x(n-(m-1)) + a_m x(n-m) = b$$

general form.

NOTE: We only have constants in front of the x's.

IF $b = 0$ we call this Homogeneous.

$$a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + \cdots + a_{m-1} x(n-(m-1)) + a_m x(n-m) = 0$$

we have a powerful soln method for these types of eqns.

<u>EX</u> Fibonacci Sequence $x(n) = x(n-1) + x(n-2)$

$$x(0) = 1$$
$$x(1) = 1$$

1. Rewrite w/ everything on LHS: $x(n) - x(n-1) - x(n-2) = 0$
   double check: ☑ constant coeficients
   ☑ homogeneous

2. Assume the soln has the form $x(n) = c\lambda^n$
   where λ is a constant.
   NOTE: This is just a guess either it will work or it wont.

3. Plug into the eqn: $x(n) = c\lambda^n$ $x(n-1) = c\lambda^{n-1}$ ...

$$c\lambda^n - c\lambda^{n-1} - c\lambda^{n-2} = 0$$

SIMPLIFY ... factor out the smallest λ^{n-m} term

$$c\lambda^{n-2}(\lambda^2 - \lambda - 1) = 0$$

so either $\underbrace{c = 0 \quad \lambda = 0}_{\text{BORING!}}$ or $\lambda^2 - \lambda - 1 = 0$

4. Solve the characteristic eqn:

$\lambda^2 - \lambda - 1 = 0$ plug into quadratic eqn $\dfrac{1 \pm \sqrt{1+4}}{2}$

$\lambda = \dfrac{1 \pm \sqrt{5}}{2}$ two soln values

5. Write the soln as a sum of possible solns.

$$x(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\lambda_1 \sim 1.618$ $\lambda_2 \sim -0.618$

These λ's are characteristic values <u>or</u> Eigenvalues.

λ_1 is the dominant eigenvalue ... long term soln is growing!

6. Solve for the C's using initial conditions.

$$x(0)=1 \quad \Rightarrow \quad C_1 + C_2 = 1$$
$$x(1)=1 \quad \Rightarrow \quad C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

TWO EQNS FOR TWO UNKNOWNS !

Solving you would find:

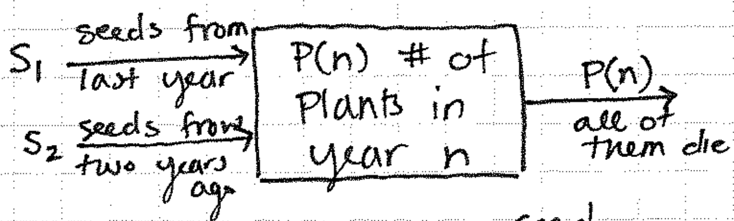
$$x(n) = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is the closed form soln for the fibonacci sequence.

Interestingly... all these irrational factors give integer values for every n.

Ex A Model for anual plants.

1. Each spring the plants produce seeds
2. The following spring some of the seeds germinate
3. Some of the seeds die
4. Some of the seeds remain dormant and germinate 2 years after being produced.
5. All grown plants die over the winter.



say γ = average # of seed ~~plants~~ each year per plant.
σ = fraction of seeds that survive a winter
α = fraction of 1 year seeds that germinate
β = fraction of 2 year seeds that germinate

$S_1 = \gamma\sigma\alpha P(n)$ and $S_2 = \gamma\sigma(1-\alpha)\sigma\beta P(n-1)$

produced, survived, germinated year 1. produced, survived winter, did not germinate, survived again, germinated year 2

$$P(n+1) - P(n) = \gamma\sigma\alpha P(n) + \gamma(1-\alpha)\sigma^2\beta P(n-1) - P(n)$$

$$P(n+1) = \gamma\sigma\alpha P(n) + \gamma(1-\alpha)\sigma^2\beta P(n-1)$$

For simplicity: $a = \gamma\sigma\alpha$ and $b = \gamma(1-\alpha)\sigma^2\beta$ constants.

$$P(n+1) = aP(n) + bP(n-1)$$

1. Rewrite: $P(n+1) - aP(n) - bP(n-1) = 0$

2. Assume soln form $P(n) = C\lambda^n$

3. Plug into eqn: $C\lambda^{n+1} - aC\lambda^n - bC\lambda^{n-1} = 0$

factor out $c\lambda^{n-1}$...

$$c\lambda^{n-1}(\lambda^2 - a\lambda - b) = 0$$

Characteristic eqn: $\lambda^2 - a\lambda - b = 0$

4. Solve Characteristic eqn:

$$\lambda = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

OR

$$\lambda = \frac{\gamma\sigma\alpha \pm \sqrt{(\gamma\sigma\alpha)^2 + 4(\gamma(1-\alpha)\sigma^2\beta)}}{2}.$$

5 Write soln as $P(n) = C_1\lambda_1^n + C_2\lambda_2^n$

INTERESTINGLY $\longrightarrow$ we can get a ton of info from the eigenvalues λ's without writing up the soln!

What if $\alpha = 1/2$ $\beta = 1/4$ $\gamma = 2$ and $\sigma = 0.8$

we find $\lambda_1 = -0.166$ and $\lambda_2 = 0.966$

$\lambda_2 \rightarrow$ dominant eigenvalue < 1

so λ_2^n is decreasing and the population is declining.

We could now experiment... Which parameters are most important in bringing λ_2 above 1? For what conditions do we avoid population collapse?

NEXT TIME Fixed Points, Stability and non constant growth rates.