Lecture 6 Math Modeling.

PROJECT 2 DUE ~~THURS~~
FINAL PROJ IDEAS DUE THURS.

Last Time: Closed Form Solns.

For simple eqns: Use Induction (AKA Find a pattern.)

(Higher order) Eqns with Constant Coeficients
+ Homogeneous: Find Eigenvalues.
Assume soln looks line: $C\lambda^n$
Plug this in and solve for λ's
Apply initial conditions to find C's

Dominant Eigenvalue ⇒ long term Behavior

$\lambda > 1$ Growth
$0 < \lambda < 1$ Decay
$\lambda < 0$ oscillations.

Lots of info from Eigenvalues.

We had the example of Plants whose seeds can germinate in either the first or second year.

FOUND
$$\lambda = \frac{\gamma\sigma\alpha \pm \sqrt{(\gamma\sigma\alpha)^2 + 4[\gamma(1-\alpha)\sigma^2\beta]}}{2}$$

if $\alpha = 0.5$ (germination year 1), $\beta = 0.25$ (germination year 2), $\gamma = 2$ (# of seeds per plant) and $\sigma = 0.8$ (fraction that survive winter).

Then $\lambda_1 = -0.166$ and $\lambda_2 = 0.966$

Homework 4 → Create a spreadsheet that can solve for λ_1 and λ_2 for any parameter values. AND finds the closed form solution.

Ex spreadsheet. if $P(0) = A$ and $P(1) = B$ $P(n) = C_1\lambda_1^n + C_2\lambda_2^n$
THEN
$$C_1 + C_2 = A$$
$$C_1\lambda_1 + C_2\lambda_2 = B$$
solve these for C_1 and C_2

ENTER PARAMETERS $\gamma, \sigma, \alpha, \beta$ AND $P(0)$, $P(1)$ A, B

↓

Use $\gamma, \sigma, \alpha, \beta$ to solve for EIGENVALUES λ_1 and λ_2

→ Use λ_1 and λ_2 ALONG WITH IC's A, B to solve for CONSTANTS C_1 and C_2

↓

soln $P(n) = C_1\lambda_1^n + C_2\lambda_2^n$

→ PLOT SOLN

TIME	POPULATION
1	$C_1\lambda_1^1 + C_2\lambda_2^1$
2	$C_1\lambda_1^2 + C_2\lambda_2^2$
3	⋮
4	
5	

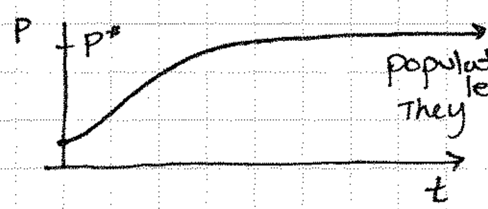
TODAY: FIXED POINTS and STABILITY.

IMPORTANT QUESTION → Do things in my dynamical system ever stop changing?

Is there ever a point where things are in perfect ballance?

We are seeking a STEADY STATE or FIXED POINT.

• Graphically:



population levels off. They are STEADY indicates a fixed point at P*

But a graph is not enough!

MATHEMATICALLY: Steady state means things dont change

$\boxed{P(n)}$ NO ARROWS.

$P(n+1) - P(n) = 0$ OR $P(n+1) = P(n) = \overline{P}$

we use this idea to Test our recurrence relations.

Ex $P(n+1) = P(n) + rP(n) + h = RP(n) + h$ where $R = (1+r)$

1. Plug in $P(n+1) = P(n) = \overline{P}$ force a steady state.

$$\overline{P} = R\overline{P} + h$$

2. Solve for $\overline{P}$: $(1-R)\overline{P} = h$ $\quad \overline{P} = \dfrac{h}{1-R} = \dfrac{h}{1-1-r} = -\dfrac{h}{r}$

3. Use the STEADY STATE soln to answer questions

$\bar{P} = -\frac{h}{r}$ is the eqn for a FIXED POINT

NOTE: P_0 does not appear!

say we measured $r = -0.028$ and want to achieve a FIXED population of $\bar{P} = 100$

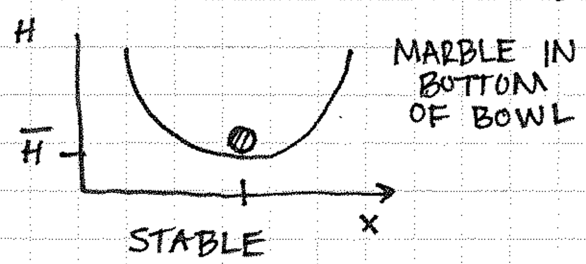
PLUG IN: $100 = \frac{-h}{-0.028}$ $\quad 100(0.028) = h \quad h = 2.8$

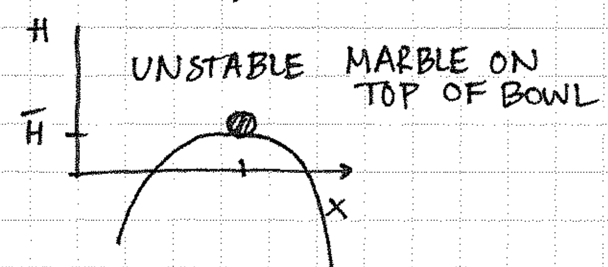
we need to add 2.8 animals a year to have a fixed point $\bar{P} = 100$.

4. Check the STABILITY of the FIXED POINT.

STABILITY MATTERS!

A FIXED POINT tells me where things are in ballance but there are different ways for this to happen.





These both have $\bar{H}$ as a ballance point, but one is STABLE and the other is UNSTABLE

A FIXED POINT is STABLE if even when we give it a small push it returns to the FIXED POINT
AKA. Solutions go TOWARD the FIXED POINT

A FIXED POINT is UNSTABLE if when we give it a small push it goes away from the FIXED POINT
AKA Solutions go AWAY from the FIXED POINT

<u>THEOREM</u> The fixed point $\bar{P}$ is stable if and only if given $P(n+1) = f(P(n))$ then

$$|f'(\bar{P})| < 1.$$

our system was $P(n+1) = RP(n) + h$

1. TAKE the DERIVATIVE of the RHS
$$f'(P(n)) = \frac{d}{dP}\left[RP+h\right] = R$$
2. PLUG IN $\overline{P}$ ... $f'(P(n)) = R$
so $f'(\overline{P}) = R$ in this case its just a constant.
3. TAKE ABSOLUTE VALUE
and check < 1.

$$|R| < 1 \Rightarrow |1+r| < 1$$
for our FIXED POINT to be STABLE.

we said $r = -0.028$

$$|1-0.028| < 1 \quad \checkmark$$
our FIXED POINT is STABLE.

SPREADSHEET EXAMPLE ...

EX What if a park manager said $r = 0.028$ and $h = 3$ ... what is our long term stable population?
[SAME RECURRENCE RELATION]

$\overline{P} = \frac{-h}{r}$ and need $|1+r| < 1$

$\overline{P} = \frac{-3}{0.028}$ and $1+r = 1.028 > 1$

we have a negative fixed point! AND it is unstable!

TOUGH LUCK PARK MANAGER!

EX Our growth rate is $r = 0.028$ and we allow for hunting $h = -2$. What is going to happen?

$$\overline{P} = -\frac{(-2)}{0.028} = \frac{2}{0.028} \sim 71.42$$

BUT $|1+r| > 1$ NOT STABLE ||

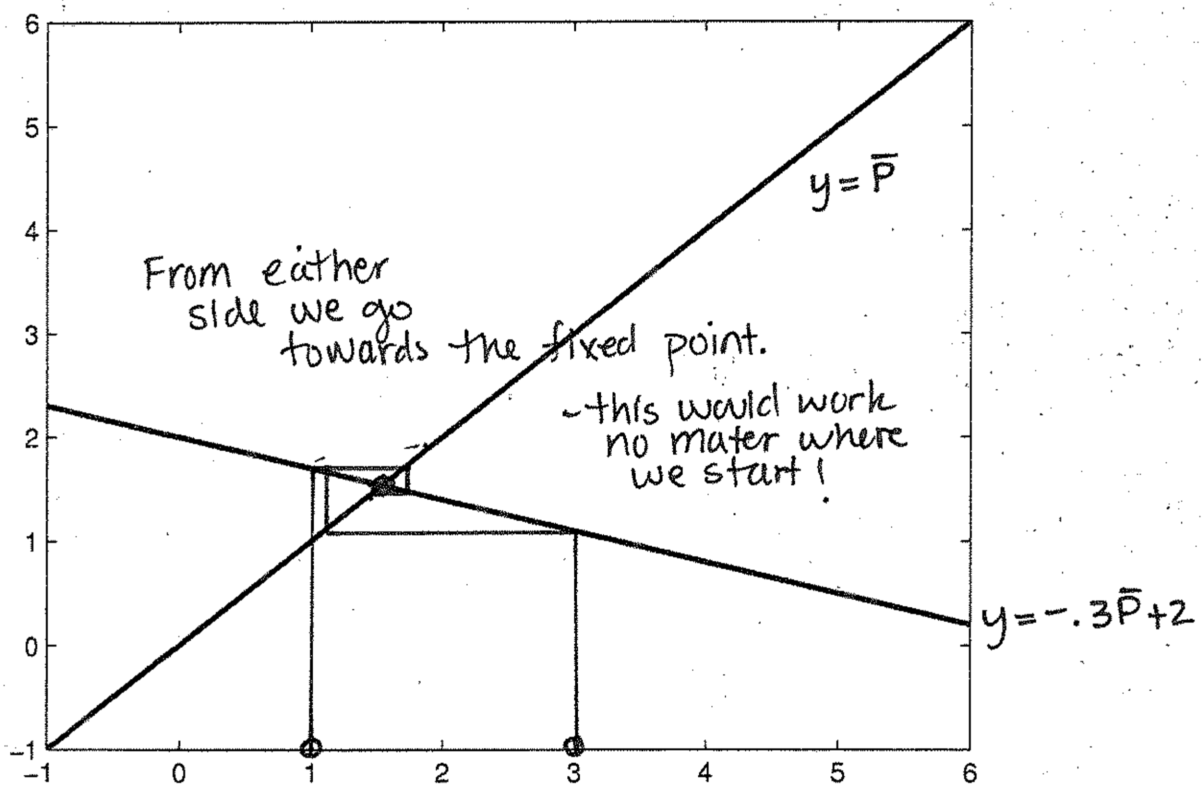
We can also think about stability by careful graphing...

THE COBWEB METHOD:

1. Graph the LEFT and RIGHT of recurrence relation on same plot
2. Points where the cross FIXED POINTS
3. Draw alternating lines to determine stability "COBWEB"

$P(n+1) = -.3P(n) + 2$

Fixed Point: $\bar{P} = -.3\bar{P} + 2$



- Where they cross is the fixed point

$\bar{P} = 1.538$

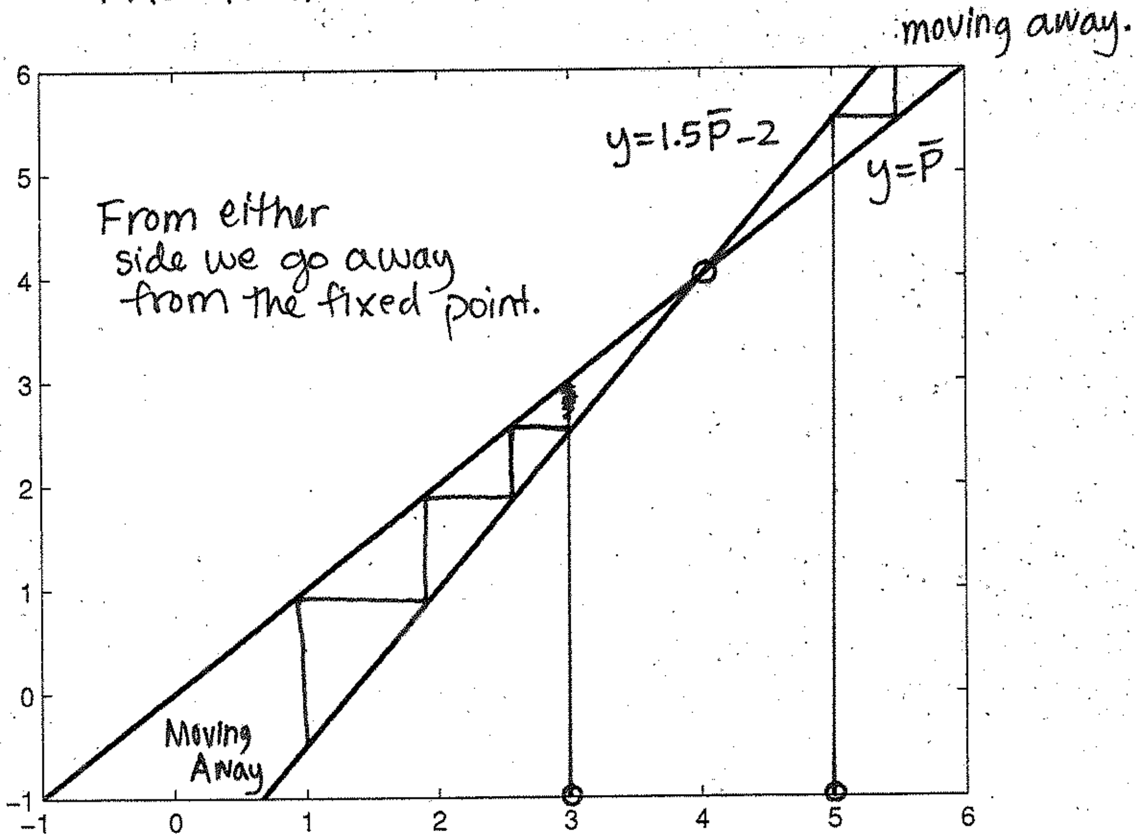
- Mathematical analysis

$(-.3P(n)+2)' = -.3 \qquad |-.3| = .3 < 1$

should be stable!

$P(n+1) = 1.5\,P(n) - 2$

Fixed Point: $\bar{P} = 1.5\bar{P} - 2$



- where they cross is the fixed point:

$\bar{P} = 4$

physically this might work but we still test stability.

- Mathematical Analysis

$(1.5P(n) - 2)' = 1.5 \qquad |1.5| = 1.5 > 1$

should <u>not</u> be stable