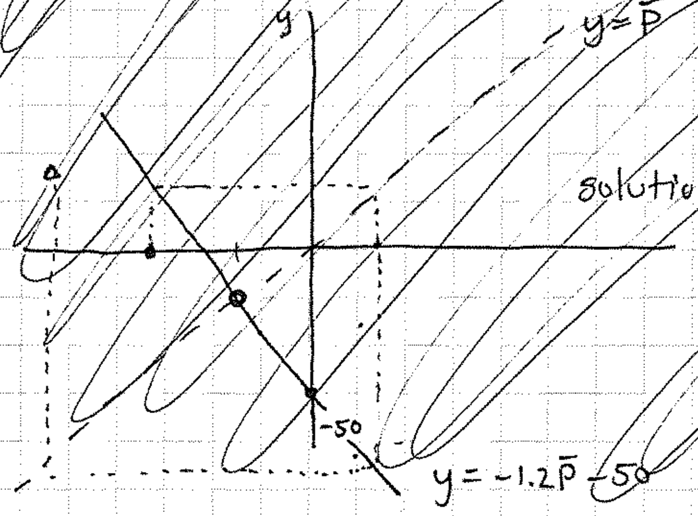
EX $P(n+1) = -1.2\,P(n) - 50$

$\bar{P} = -1.2\,\bar{P} - 50$

plot $y = \bar{P}$ and $y = -1.2\bar{P} - 50$



solution is moving away → unstable.

VARIABLE GROWTH RATES.

Many models do not have constant growth rates

"In 1977 there were 37 Elvis impersonators in the world. In 1993 there were 48,000. At this rate, by the year 2010, one out of every three people will be an Elvis impersonator."

Problem w/ Constant Growth – even if animals are piled 12 deep it may still predict growth.

These models are valid for a limited # of problems!

Using variable growth rates will help!

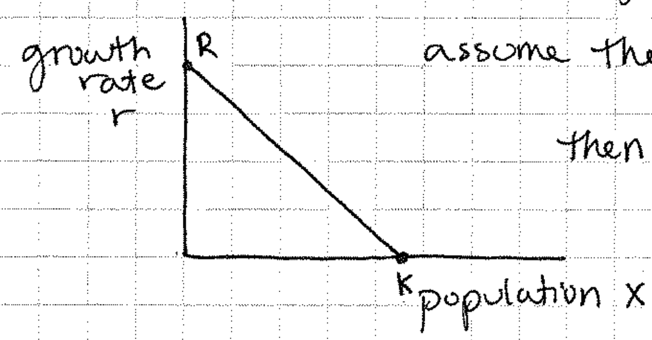
$$X(n+1) = X(n) + \underbrace{r(X(n))}_{\text{the growth rate is no longer constant! It depends on the \# of animals.}}\,X(n)$$

What should our variable growth rate look like?

A POPULAR ASSUMPTION: as the population increases, the growth rate decreases because of a combination of increased deaths or decreased births.

CARRYING CAPACITY – the maximum population that can be supported.

so when the population = carrying capacity the growth rate = 0
and when the population is smaller the growth rate = maximum.



assume the dependence is linear...

then $r(x) = -\frac{R}{K}x + R$

Plug this in...

$$X(n+1) = X(n) + \left(R - \frac{R}{K}X(n)\right)X(n)$$

$$= X(n) + R\left(1 - \frac{X(n)}{K}\right)X(n)$$

$$\boxed{X(n+1) = X(n) + R\left(1 - \frac{X(n)}{K}\right)X(n)}$$

THIS IS THE LOGISTIC EQN.

A VERY IMPORTANT MODEL!!
NONLINEAR!

Now lets find fixed points and analyze stability...

FIXED POINTS $P(n+1) = P(n) = \bar{P}$ the population stops changing

sub this into logistic eqn. $\bar{P} = \bar{P} + R\left(1 - \frac{\bar{P}}{K}\right)\bar{P}$

now solve for $\bar{P}$ $\quad \bar{P} - \bar{P} = R\left(1 - \frac{\bar{P}}{K}\right)\bar{P}$

$0 = R\left(1 - \frac{\bar{P}}{K}\right)\bar{P}$ so either $\boxed{\bar{P} = 0}$ or $\boxed{\bar{P} = K}$ two fixed points!

NEXT STABILITY $\quad f(\bar{P}) = \bar{P} + R\left(1 - \frac{\bar{P}}{K}\right)\bar{P}$

$$= \bar{P} + R\left(\bar{P} - \frac{\bar{P}^2}{K}\right) = (1+R)\bar{P} - \frac{R}{K}\bar{P}^2$$

then $\underbrace{f'(\bar{P}) = (1+R) - \frac{2R}{K}\bar{P}}$

This still has a $\bar{P}$ so we need to check each fixed point.

$\bar{P} = 0$ then $f'(0) = 1 + R$

so if $|f'(\bar{P})| = |1+R| < 1$ then $\bar{P} = 0$ is stable

$|1+R| < 1 \Rightarrow -1 < 1+R < 1$

and $\boxed{-2 < R < 0}$

if our maximum growth rate is in this range $\bar{P}=0$ is stable.

$\bar{P} = K \quad f'(K) = (1+R) - 2R = 1 - R$

so if $|f'(\bar{P})| = |1-R| < 1$ then $\bar{P} = K$ is stable

$|1-R| < 1 \Rightarrow -1 < 1-R < 1$

and $-2 < -R < 0$

$+2 > R > 0$

$\boxed{0 < R < 2}$

if our maximum growth rate is in this range $\bar{P} = K$ is stable.

Experiment on Excel.

Math 231
Logistic Models and Spreadsheets

Goal: Write a spreadsheet program for the Logistic Equation.

On a separate piece of paper write down the Logistic Recurrence relation. What are your variables? What are your parameters? Draw a compartmental diagram for the Logistic Equation.

Now open Excel (or another spreadsheet program). Start by listing all of your parameters at the top of your spreadsheet. To start let the initial population be 25, the maximum growth rate be 0.5 and the carrying capacity be 120. Build your model and make a graph plotting population vs. time.

Next we will test to see if the model is sensitive to any of our parameters.

- Keeping the carrying capacity and growth rate the same, change the initial population. Vary it from above the carrying capacity down to zero. Is your model sensitive to initial population? Does your answer make sense?
- Keeping the initial population and growth rate the same, change the carrying capacity. Is your model sensitive to the carrying capacity? Does your answer make sense?
- Keeping the initial population and carrying capacity the same, change the maximum growth rate.
  - Vary R between $-2 < R < 0$. Our analysis said that $\bar{P} = 0$ is the stable fixed point. Do you see this in your model?
  - Vary R between $0 < R < 2$. Our analysis said that $\bar{P} = K$ is the stable fixed point. Do you see this in your model?
  - What happens when you are outside these ranges? Let $R = 2$, $R = -2$, or let R be way outside these ranges. What do you see? Try to describe in words what your model is telling you. We will talk about this at the end of class.

Logistic Equation

One often looks toward physical systems to find chaos, but it also exhibits itself in biology. Biologists had been studying the variability in populations of various species and they found an equation that predicted animal populations reasonably well. This equation was a simple quadratic equation called the logistic difference equation. On the surface, one would not expect this equation to provide the fantastically complex and chaotic behavior that it exhibits.

The logistic difference equation is given by

$$x_{n+1} = rx_n(1 - x_n)$$

where *r* is the so-called driving parameter. The equation is used in the following manner. Start with a fixed value of the driving parameter, *r*, and an initial value of x_0. One then runs the equation recursively, obtaining $x_1, x_2, \ldots x_n$. For low values of *r*, x_n (as *n* goes to infinity) eventually converges to a single number. In biology, this number (x_n as *n* approaches infinity) represents the population of the species.

It is when the driving parameter, *r*, is slowly turned up that interesting things happen. When $r = 3.0$, x_n no longer converges — it oscillates between two values. This characteristic change in behavior is called a bifurcation. Turn up the driving parameter even further and x_n oscillates between not two, but four values. As one continues to increase the driving parameter, x_n goes through bifurcations of period eight, then sixteen, then chaos! When the value of the driving parameter *r* equals 3.57, x_n neither converges or oscillates — its value becomes completely random. For values of *r* larger than 3.57, the behavior is largely chaotic. However, there is a particular value of *r* where the sequence again oscillates with period of three. The bifurcations then begin again with period 6, 12, 24, then back to chaos. In fact it was discovered in James Yorke's famous paper "Period Three Implies Chaos." that any sequence with a period of three will display regular cycles of every other period as well as exhibiting chaotic cycles.

The bifurcation diagram of the logistic difference equation is shown below: