Math Modeling Lecture 8

Last Time: Variable Growth rates:

$$X(n+1) = x(n) + r(x(n))\,x(n)$$

1. Assume growth / death or other rate parameters are variable.
   Depend on TIME, DISTANCE, POPULATION etc.
2. Describe in words how those rates vary (Draw pictures or graphs)
3. Come up with eqns that make sence with your description in part 3.
4. Sub the variable rates into your recurrence relation.

ONE EXAMPLE: The Logistic Equation

$$P(n+1) = P(n) + \underbrace{R}_{\text{max growth}}\left(1 - \frac{P(n)}{\underbrace{K}_{\text{carrying capacity}}}\right)P(n)$$

TODAY: SYSTEMS OF RECURRENCE RELATIONS.

- when we have two or more time dependent variables
- Interaction between speecies, companies, countries, religions ... etc.
- competition / cooperation

Our ideas from past models still apply:

FIXED POINTS – points where neither variable changes
STABILITY – solns go toward stable fixed points and away from unstable fixed points
EIGENVALUES – for linear systems solve using matrix methods – characteristic values that answer the question will the population grow or decay?
– for nonlinear systems it is more complicated
NONLINEAR DYNAMICS AND CHAOS

Ex Lets build a Host - Parasite Model.

General Info: Experimentalists say that the notice some VERY complicated behavior when studying hosts and parasites. The populations seem to oscillate over time.

Goal: Model this using simple assumptions - do we see this oscillation in population?

BRAINSTORM: Assume we are dealing with two types of insect that grow in discrete cycles.

Parasite - attaches to host - weakens host
- too many parasites the host dies
- if the host species dies parasite dies.

Host - parasite weakens them - nothing they can really do about it.

Assume without parasites (predators) the host (prey) population shows logistic growth.

with the parasite some of the host population is removed.

$$\xrightarrow{r(H(n))H(n)} \boxed{\begin{array}{c} H(n) \\ \text{\# of Host Animals} \end{array}} \longrightarrow \text{Predator (Parasite) Dammage.}$$

$$r(H(n)) = R\left(1 - \frac{H(n)}{K}\right)$$

To model predator Dammage we assume that for damage to occur the two species must interact.

$$\text{Damage} = \underbrace{E}_{\text{Efficiency of the predator}} H(n) \underbrace{P(n)}_{\text{\# of Parasites.}}$$

Interaction is proportional to the product of the two populations.

The product approximates the likelihood of encounter assuming the two animals move about randomly - Law of mass action from Physical Chemistry.

Problem... parasites never get full... constant predidation.

$$R\left(1-\frac{H(n)}{K}\right)H(n) \xrightarrow[\text{Logistic Growth}]{} \boxed{\begin{array}{c} H(n) \\ \#\text{ of host animals}\end{array}} \xrightarrow[\text{damage from parasite.}]{EH(n)P(n)}$$

NOW: How do the parasites change

$$\xrightarrow[\text{hosts into growth of parasites.}]{\text{They turn encounters w/}} \boxed{\begin{array}{c} P(n) \\ \#\text{ of Parasites}\end{array}} \xrightarrow{P(n)}$$

no real death rate the whole population dies @ each time step.

with no hosts they totally die out.

$$\text{GROWTH} = Q\,H(n)P(n)$$

Q: Efficiency of turning hosts into parasites.

Write Equations for each population:

$$H(n+1)-H(n) = R\left(1-\frac{H(n)}{K}\right)H(n) - E\,H(n)P(n)$$

$$P(n+1)-P(n) = Q\,H(n)P(n) - P(n)$$

OUR SYSTEM OF RECURRENCE RELATIONS

$$\begin{cases} H(n+1) = H(n) + R\left(1-\dfrac{H(n)}{K}\right)H(n) - E\,H(n)P(n) \\ P(n+1) = Q\,H(n)P(n) \\ \\ H(0)=H_0 \qquad P(0)=P_0 \end{cases}$$

NON LINEAR !

FIXED POINTS:

$$H(n+1) = H(n) = \overline{H}$$
$$P(n+1) = P(n) = \overline{P}$$

NOTE: our analysis only works for first order systems. In nonlinear dynamics / differential eqns we show ANY higher order system can be reduced to a set of first order!

looking for where BOTH stop changing... a fixed point of the system.

PLUG IN:

$$\left.\begin{array}{l} \overline{H} = \overline{H} + R\left(1-\dfrac{\overline{H}}{K}\right)\overline{H} - E\,\overline{H}\,\overline{P} \\ \overline{P} = Q\,\overline{H}\,\overline{P} \end{array}\right\} \text{Solve for } \overline{H} \text{ and } \overline{P}$$

$$0 = \left(\left(R - \frac{R\bar{H}}{K}\right) - E\bar{P}\right)\bar{H} \qquad \text{OR} \qquad 0 = \left(R - \frac{R\bar{H}}{K} - E\bar{P}\right)\bar{H}$$

$$0 = \left(Q\bar{H} - 1\right)\bar{P} \qquad\qquad 0 = \left(Q\bar{H} - 1\right)\bar{P}$$

CASES:

$$\left.\begin{array}{l} R - \frac{R\bar{H}}{K} - E\bar{P} = 0 \quad \text{OR} \quad \bar{H} = 0 \end{array}\right] \quad \text{AND}$$

$$\left.\begin{array}{l} Q\bar{H} - 1 = 0 \quad \text{OR} \quad \bar{P} = 0 \end{array}\right]$$

From Top, never works!

if $\bar{H} = 0$ then either $-1 = 0$ or $\bar{P} = 0$ $\Rightarrow$ $\bar{H} = 0 \quad \bar{P} = 0$

Plug into Bottom

From Bottom

if $\bar{P} = 0$ then $R - \frac{R\bar{H}}{K} = 0 \Rightarrow \bar{H} = K$ $\Rightarrow$ $\bar{H} = K \quad \bar{P} = 0$

From Bottom

if $\bar{H} = 1/Q$ then $R - \frac{R}{QK} - E\bar{P} = 0$ $\Rightarrow$ $\bar{H} = 1/Q \quad \bar{P} = \frac{R}{E}\left(1 - \frac{1}{QK}\right)$

$$\bar{P} = \frac{R}{E}\left(1 - \frac{1}{QK}\right)$$

What do these mean? $(\bar{H}, \bar{P})$

$(0,0)$ – no animals are alive

$(K,0)$ – no parisites the host is at carying capacity

$\left(1/Q, \frac{R}{E}\left(1 - 1/QK\right)\right)$ – A perfect ballance! Both can survive.

STABILITY IS HARDER TO SHOW $\longrightarrow$ has to do with the eigen values of the Jacobian matrix for the associated linearized system.

<u>BUT</u> we can test stability in our spreadsheet.

Example Spreadsheet: predator_prey.ods.

$R = 1.5$

$K = 100$

$\left.\begin{array}{l} E = .02 \\ Q = .02 \end{array}\right\}$ these should be pretty small!

UNITS $E = \%\ \text{change}/\text{Parasite}$ $\quad Q = \%\ \text{growth}/\text{Host}.$

To Numerically (Spreadsheet) test stability:

1. Plug in the Fixed Points Exactly (Initial Conditions)
   - this should show that the populations remain the same
2. Make a small change in one or both initial conditions and see what happens.
   - Solution goes away from fixed point then UNSTABLE
   - Soln goes toward fixed point then STABLE
   - Oscillates around fixed point... STABLE w/ LIMIT CYCLE OSCILLATIONS... harder to prove.
3. Discuss the stability based on your spreadsheet or model predictions.

~~HW 7 math models for Love. Due in 1 week after fall break.~~