

Math Modeling Lecture 8

Last Time: Variable Growth rates:

$$X(n+1) = X(n) + r(x(n)) X(n)$$

1. Assume growth/death or other ^{rate} parameters are variable.

Depend on TIME, DISTANCE, POPULATION etc.

2. Describe in words how those rates vary (Draw pictures or graphs)
3. Come up with eqns that make sense with your description in part 3.
4. Sub the variable rates into your recurrence relation.

ONE EXAMPLE: The Logistic Equation

$$P(n+1) = P(n) + R \left(1 - \frac{P(n)}{K} \right) P(n)$$

max growth carrying capacity.

TODAY: SYSTEMS OF RECURRENCE RELATIONS.

- when we have two or more time dependent variables
- Interaction between species, companies, countries, religions ... etc.
- competition/cooperation

Our ideas from past models still apply:

FIXED POINTS - points where neither variable changes

STABILITY - solns go toward stable fixed points and away from unstable fixed points

EIGENVALUES - for linear systems solve using matrix methods - characteristic values that answer the question will the population grow or decay?

- for nonlinear systems it is more complicated

NONLINEAR DYNAMICS AND CHAOS

Ex Lets build a Host - Parasite Model.

General Info: Experimentalists say that the notice some VERY complicated behavior when studying hosts and parasites. The populations seem to oscillate over time.

Goal: Model this using simple assumptions - do we see this oscillation in population?

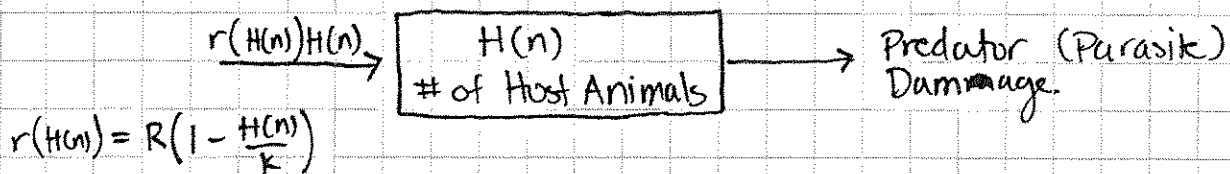
BRAINSTORM: Assume we are dealing with two types of insect that grow in discrete cycles.

Parasite - attaches to host - weakens host
 - too many parasites the host dies
 - if the host species dies parasite dies.

Host - parasite weakens them - nothing they can really do about it.

Assume with out parasites the host population shows logistic growth.

with the parasite some of the host population is removed.



To model predator Damage we assume that for damage to occur the two species must interact.

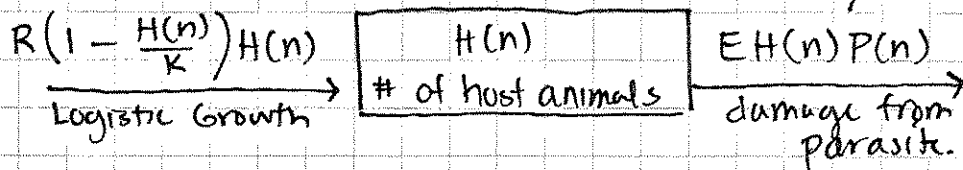
$$\text{Damage} = E H(n) P(n)$$

Efficiency of the predator

of parasites.

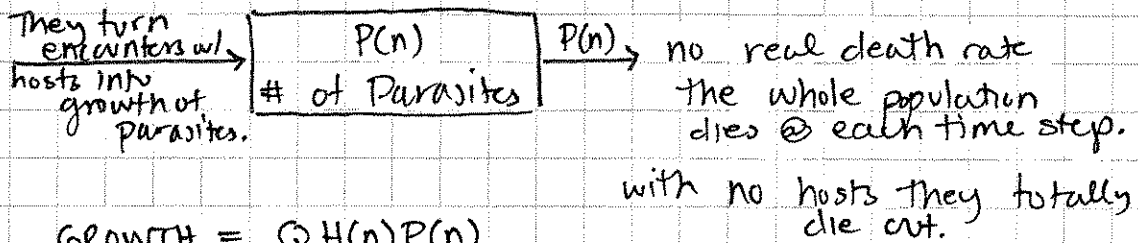
Interaction is proportional to the product of the two populations.

The product approximates the likelihood of encounter assuming the two animals move about randomly - Law of mass action from Physical Chemistry.



3. Problem... parasites never get full... constant predation.

NOW: How do the parasites change



$$\text{GROWTH} = Q H(n) P(n)$$

Efficiency of turning hosts into parasites.

Write Equations for each population:

$$H(n+1) - H(n) = R\left(1 - \frac{H(n)}{K}\right)H(n) - E H(n) P(n)$$

$$P(n+1) - P(n) = Q H(n) P(n) - P(n)$$

OUR SYSTEM OF RECURRENCE RELATIONS

$$\begin{cases} H(n+1) = H(n) + R\left(1 - \frac{H(n)}{K}\right)H(n) - E H(n) P(n) \\ P(n+1) = Q H(n) P(n) \end{cases}$$

NON LINEAR !

$$H(0) = H_0 \quad P(0) = P_0$$

FIXED POINTS:

$$\begin{aligned} H(n+1) &= H(n) = \bar{H} \\ P(n+1) &= P(n) = \bar{P} \end{aligned}$$

NOTE: Our analysis only works for first order systems. In nonlinear dynamics / differential eqns we show ANY higher order system can be reduced to a set of first order!

looking for where BOTH stop changing... a fixed point of the system.

PLUG IN:

$$\left. \begin{aligned} \bar{H} &= \bar{H} + R\left(1 - \frac{\bar{H}}{K}\right)\bar{H} - E \bar{H} \bar{P} \\ \bar{P} &= Q \bar{H} \bar{P} \end{aligned} \right\} \text{solve for } \bar{H} \text{ and } \bar{P}$$

$$0 = \left(\left(R - \frac{R\bar{H}}{K} \right) - E\bar{P} \right) \bar{H} \quad \text{OR} \quad 0 = \left(R - \frac{R\bar{H}}{K} - E\bar{P} \right) \bar{H}$$

$$0 = (Q\bar{H} - 1) \bar{P} \quad \text{OR} \quad 0 = (Q\bar{H} - 1) \bar{P}$$

CASES: $R - \frac{R\bar{H}}{K} - E\bar{P} = 0$ OR $\bar{H} = 0$] AND

$Q\bar{H} - 1 = 0$ OR $\bar{P} = 0$]

From Top
if $\bar{H} = 0$ then either $-1 = 0$ or $\bar{P} = 0$ $\Rightarrow \bar{H} = 0 \quad \bar{P} = 0$
never works!
Plug into Bottom

From Bottom
if $\bar{P} = 0$ then $R - \frac{R\bar{H}}{K} = 0 \Rightarrow \bar{H} = K \Rightarrow \bar{H} = K \quad \bar{P} = 0$

From Bottom
if $\bar{H} = 1/Q$ then $R - \frac{R}{QK} - E\bar{P} = 0 \Rightarrow \bar{H} = 1/Q \quad \bar{P} = \frac{R}{E} \left(1 - \frac{1}{QK} \right)$
 $\bar{P} = \frac{R}{E} \left(1 - \frac{1}{QK} \right)$

What do these mean? $(\bar{H}, \bar{P})$

$(0, 0)$ - no animals are alive

$(K, 0)$ - no parasites the host is at carrying capacity

$(1/Q, \frac{R}{E} (1 - 1/QK))$ - A perfect balance! Both can survive.

STABILITY IS HARDER TO SHOW $\rightarrow$ has to do with the eigen values of the Jacobian matrix for the associated linearized system.

BUT we can test stability in our spreadsheet.

Example Spreadsheet: predator-prey.xls.

$$R = 1.5$$

$$K = 100$$

$$E = .02$$

$$Q = .02$$

} these should be pretty small!

UNITS

$E = \% \text{ change} / \text{Parasite}$

$Q = \% \text{ growth} / \text{Host.}$

To Numerically (Spreadsheet) test stability:

1. Plug in the Fixed Points Exactly (Initial Conditions)
 - this should show that the populations remain the same.
2. Make a small change in one or both initial conditions and see what happens.
 - Solution goes away from fixed point then UNSTABLE
 - Soln goes toward fixed point then STABLE
 - Oscillates around fixed point...
STABLE w/ LIMIT CYCLE OSCILLATIONS...
harder to prove.
3. Discuss the stability based on your spreadsheet or model predictions.

~~HW 7 math models for Love. Due in 1 week~~
after fall break.