Last time – different kinds of fixed points
– all depends on the exponent
* eigenvalues
* eigenvectors.

EIGENVALUES...

remember we are trying to solve $\dot{\vec{x}} = A\vec{x}$

$\vec{x}$ – soln matrix $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Seek solns of the form $e^{\lambda t}$

then $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

so we look for $\lambda \vec{x} = A\vec{x}$

* this means that $0 = A\vec{x} - \lambda\vec{x}$

$0 = (A - \lambda I)\vec{x}$

I – Identity matrix

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For a soln to exist

$\det(A - \lambda I) = 0$

look for λ's that make this happen

In General $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

solve for λ.

$$ad - (a+d)\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \text{quadratic formula}$$

let $\tau = a + d$ — TRACE of A add along diagonal

$\Delta = ad - bc$ — Determinant of A.

$$\lambda^2 - \tau\lambda + \Delta = 0 \quad \text{or} \quad \boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

in our 2-D system.

CASES

$\tau^2 - 4\Delta > 0$ real distinct λ. — Nodes or Saddles.

$\tau^2 - 4\Delta = 0$ repeated not distinct λ. — Special Cases.

$\tau^2 - 4\Delta < 0$ complex or imaginary λ — Centers or Spirals.

Just from looking at A we can see what is happening at the fixed point.

EIGENVECTORS
after we find eigenvalues we solve for eigenvectors.

$(A - \lambda I)\vec{x} = 0$ plug in the eigenvalue and solve for $\vec{x}$.

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

not a unique solution!

SOLVE THE SYSTEM $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

only works for the linear systems — lots of special cases — TAKE ODE'S and LINEAR!!

EX $\dot{x} = x + y$
$\dot{y} = 4x - 2y$
$(x_0, y_0) = (2, 3)$

1 Write in Matrix Form
2. Sketch the Phase Portrait
3. Solve.

IN MATRIX FORM $\dot{\vec{x}} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{x}$ so $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

SKETCH — need eigenvalues! then eigenvectors!

Trace — $\tau = 1 + (-2) = -1$

Determinant — $\Delta = ad - bc = 1(-2) - (1)(4) = -2 - 4 = -6.$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{25} = -\frac{1}{2} \pm \frac{5}{2}$$

$\lambda_1 = \frac{4}{2} = 2$ $\lambda_2 = -\frac{6}{2} = -3$

one positive and one negative exponent!
This is some sort of saddle point.

Find eigenvectors — do the calculation for *each* eigenvalue.

Choose $\lambda_1 = 2$ and plug in: $(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -x_1 + x_2 = 0 \\ 4x_1 - 4x_2 = 0 \end{matrix} \xrightarrow{\text{both}} x_1 = x_2$$

let $x_1 = 1$ then $x_2 = 1$ and $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(this is a choice... we can choose anything except zero!

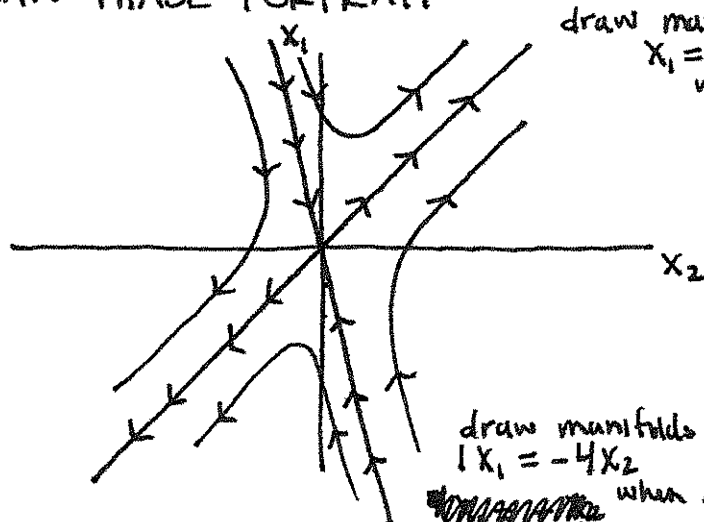
Choose $\lambda_2 = -3$ and plug in: $(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 4x_1 + x_2 = 0 \\ 4x_1 + x_2 = 0 \end{matrix} \xrightarrow{\text{both}} 4x_1 = -x_2$$

let $x_1 = 1$ then $x_2 = -4$ and $\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

DRAW PHASE PORTRAIT



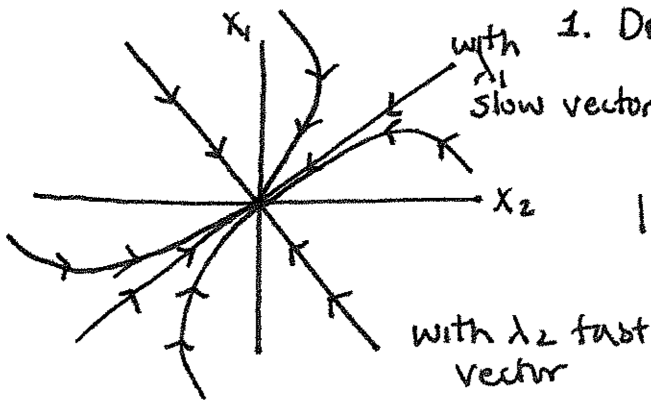
draw manifolds $x_1 = x_2$ when $\lambda = 2$

last draw trajectories.

draw manifolds $1x_1 = -4x_2$ when $\lambda = -3$

OTHER CASES:

A. Two negative eigenvalue $\lambda_2 < \lambda_1 < 0$ STABLE NODE



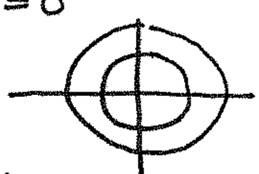
1. Draw Eigenvectors.
Instead of pos-neg line the saddle we now have slow fast.

$|\lambda_2| > |\lambda_1|$
so λ_2 decays faster!

Approach the node along the slow vector.

Two positive eigenvalues.
if $0 < \lambda_1 < \lambda_2$ UNSTABLE NODE

B. Centers if $\tau^2 - 4\Delta = 0$ and $\tau = 0$ λ_1 and λ_2 are pure imaginary.



center.

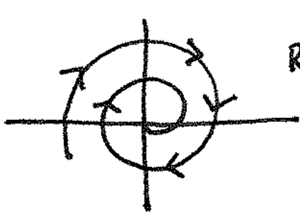
To determine the direction of rotation ... calculate some flows. plug into $\dot{x}_1$ and $\dot{x}_2$ like we did originally

C. Spirals if $\tau - 4\Delta < 0$ and $\tau \neq 0$ λ_1 and λ_2 are complex.

ex $\lambda = 1 \pm 3i$

$Re\{\lambda\} = 1$ $Im\{\lambda\} = 3$

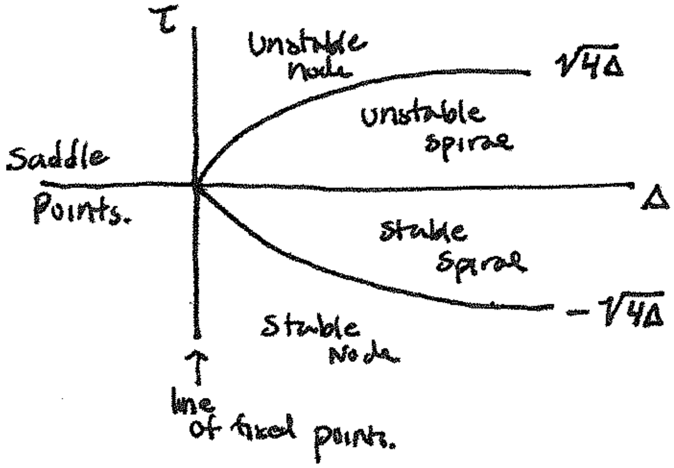
The Real part tells us growth or decay stable or unstable.



again for the direction of the spiral plug into $\dot{x}_1$ and $\dot{x}_2$

D. $\lambda_1 = \lambda_2$ when $\tau - 4\Delta = 0$ special cases...

FABULOUS DIAGRAM TO SUMMARIZE THIS...



$\tau^2 - 4\Delta = 0$ is important
plot $\tau = \pm\sqrt{4\Delta}$

YOU TRY... $\dot{\vec{x}} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Aside #1 Write down the equations:

$$\dot{x}_1 = 2x_1 + x_2$$
$$\dot{x}_2 = 3x_1 + 4x_2$$

Trace and Determinant

$\tau = 6$ and $\Delta = 5$

According to the diagram... expect unstable nodes because $\tau > \sqrt{4\Delta}$
$6 > \sqrt{20}$

Eigenvals and vectors should support your prediction

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{6}{2} \pm \frac{1}{2}\sqrt{16} = 3 \pm 2$$

$$\boxed{\lambda_1 = 1 \qquad \lambda_2 = 5}$$

choose $\lambda_1 = 1$

$$(A - \lambda I)\vec{x} = \begin{bmatrix} 2-1 & 1 \\ 3 & 4-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} x_1 + x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{matrix} \longrightarrow x_1 = -x_2$$

let $x_1 = 1$ then $x_2 = -1$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

choose $\lambda_2 = 5$

$$(A - \lambda I)\vec{x} = \begin{bmatrix} 2-5 & 1 \\ 3 & 4-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -3x_1 + x_2 &= 0 \\ 3x_1 - x_2 &= 0 \end{aligned} \longrightarrow 3x_1 = x_2$$

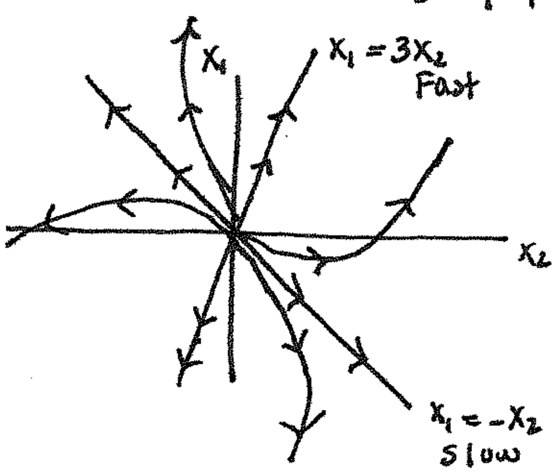
let $x_1 = 1$ then $x_2 = 3$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Phase Portrait.

$\lambda_1 = 1 \quad \vec{x}_1 = [1, -1]^T$

$x_1 = -x_2$ Slow vector

$\lambda_2 = 5 \quad \vec{x}_2 = [1, 3]^T \quad x_1 = 3x_2$ Fast vector.



close the the fixed point trajectory follows the slow vector.

Two unstable manifolds.