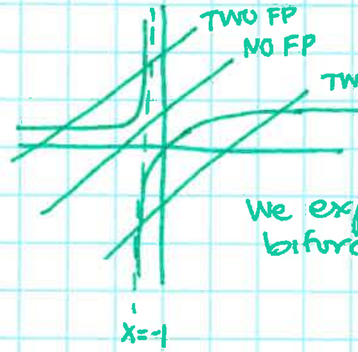
Saddle Node #4 $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$

FIND FP $r + \frac{1}{2}x - \frac{x}{1+x} = 0$

PLOT $r + \frac{1}{2}x = \frac{x}{1+x}$



We expect two bifurcation points.

We can solve this!

$$r(1+x) + \tfrac{1}{2}x(1+x) - x = 0$$
$$r + rx + \tfrac{1}{2}x + \tfrac{1}{2}x^2 - x = 0$$
$$2r + 2rx + x + x^2 - 2x = 0$$
$$+x^2 + (2r-1)x + 2r = 0$$
$$x^2 + (2r-1)x + 2r = 0$$

QUADRATIC

Fixed points. $x^* = \dfrac{-(2r-1) \pm \sqrt{(2r-1)^2 - 8r}}{2}$

CASES $(2r-1)^2 - 8r < 0$

$(2r-1)^2 - 8r = 0$ —— solve for r $\quad 4r^2 - 4r + 1 - 8r = 0$

$(2r-1)^2 - 8r > 1$

$$4r^2 - 12r + 1 = 0$$
$$r^2 - 3r + 1/4 = 0$$

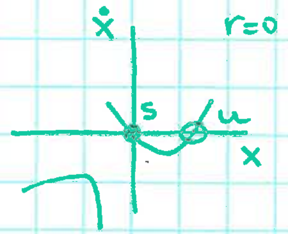
so

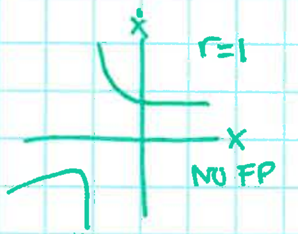
$$r = \frac{3 \pm \sqrt{9-1}}{2} = \frac{3}{2} \pm \frac{\sqrt{8}}{2} = \frac{3}{2} \pm \sqrt{2}$$

Bifurcation points $r_c = \frac{3}{2} \pm \sqrt{2}$

When

$r < \frac{3}{2} - \sqrt{2} \sim 0.086$ $\qquad 0.086 < r < 2.914$ $\qquad r > \frac{3}{2} + \sqrt{2} \sim 2.914$





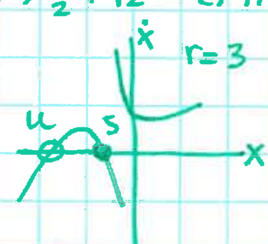
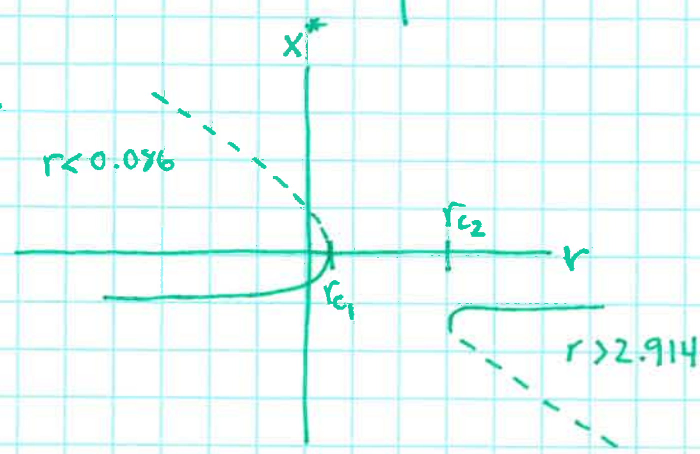


Diagram. plot x^* from eqn above.



Linear Stability.

need $f'(x) = \frac{1}{2} \cancel{+} (1+x)^{-1} + x(1+x)^{-2}$

$$= \frac{1}{2} - \frac{1}{(1+x)^2}$$

GOAL – test the stability of fixed points... so fp must exist and then we need to plug them in.

Remember the cases.

$r < 0.086$ and $r > 2.914$

choose $r=0$

then $x^* = \frac{1 \pm \sqrt{1}}{2} = \frac{1}{2} \pm \frac{1}{2}$

$x^* = 1, 0$

TOP curve, BOT curve

$f'(1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$ unstable

$f'(0) = \frac{1}{2} - 1 = -\frac{1}{2} < 0$ stable.

This matches our graph!

← Between → NO FP.

choose $r=3$

then $x^* = \frac{-(6-1) \pm \sqrt{(6-1)^2 - 24}}{2}$

$= \frac{-5 \pm \sqrt{25-24}}{2}$

$= -\frac{5}{2} \pm \frac{1}{2} = -3, -2$

BOT, TOP

$f'(-2) = \frac{1}{2} - 1 = -\frac{1}{2} < 0$ stable

$f'(-3) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$ unstable

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OR as when is $\frac{1}{2} - \frac{1}{(1+x)^2} < 0$ vs $\frac{1}{2} - \frac{1}{(1+x)^2} > 0$ GRAPH

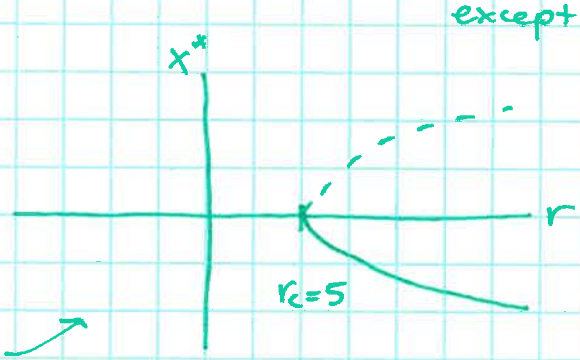
$\frac{1}{2} - \frac{1}{(1+x)^2} = 0 \qquad (1+x)^2 = 2 \qquad 1+x = \pm\sqrt{2} \qquad x = -1 \pm \sqrt{2}$

when $-1-\sqrt{2} < x < -1+\sqrt{2}$

in this range stable.

TAYLOR SERIES... we can turn any function into polynomials... like magic! except it's LOGIC :)

$$\dot{x} = 5 - re^{-x^2}$$



most people got this

SADDLE NODE at $r_c = 5$.

but e^{-x^2} is a complicated function... lets support our work with a TAYLOR SERIES analyses.

Expand $f(x) = e^{-x^2}$ near $a = n$

1 need derivatives...

$$f'(x) = -2xe^{-x^2}$$
$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$$
$$f'''(x) = 4xe^{-x^2} + 8xe^{-x^2} - 8x^3e^{-x^2}$$
$$= 12xe^{-x^2} - 8x^3e^{-x^2}$$

2. Formula

$$f(x) \sim f(n) + f'(n)(x-n) + \frac{f''(n)}{2}(x-n)^2 + \frac{f'''(n)}{3!}(x-n)^3 + \dots$$

plugging in $n=0$ is Expanding about ZERO

most memorized series are the McLaurin when $n=0$

PLUG IN THE POINT ABOUT WHICH YOU ARE EXPANDING!

we are zooming into the x-value during bifurcation $r_c = 5$

$$f(x) \sim f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \dots$$

$$= 1 + (0)(x-0) - 1(x-0)^2 + (0)(x-0)^3 + \dots = 1 - x^2 + \dots$$

3. Back to problem: $\dot{x} = 5 - (r - rx^2 + \dots) = (5-r) - rx^2 + \dots$

Form of saddle node

$r=5$ bifurcation point.

What if instead we expand about $n=1$?

$f(1) = e^{-1}$
$f'(1) = -2e^{-1}$
$f''(1) = 2e^{-1}$
$f'''(1) = 4e^{-1}$

Must plug into the function! not just the $(x-a)$ part.

$$f(x) \sim e^{-1} - 2e^{-1}(x-1) + \frac{2e^{-1}}{2}(x-1)^2 + \frac{4e^{-1}(x-1)^3}{6} + \cdots$$

NON DIMENSIONALIZATION.

$$\dot{x} = -ax + bx^2 + x^4$$

TWO parameters a, b ... could analyze BOTH or try a non-dimensionalization.

let $\xi = \frac{x}{x_0}$ and $\tau = \frac{t}{t_0}$ $\qquad x(t) \rightsquigarrow \xi(\tau)$

$x = x_0 \xi$

Then the ODE becomes...

plug in $\xi(\tau)$ $\qquad \frac{d}{dt}\left[x_0 \xi(\tau)\right] = -a x_0 \xi + b x_0^2 \xi^2 + x_0^4 \xi^4$

deal with d/dt $\qquad \frac{d}{dt}\xi(\tau) = \frac{d\xi}{d\tau}\frac{d\tau}{dt} = \frac{1}{t_0}\frac{d\xi}{d\tau} = \frac{\dot{\xi}}{t_0}$

so $\qquad \frac{x_0}{t_0}\dot{\xi} = -a x_0 \xi + b x_0^2 \xi^2 + x_0^4 \xi^4$

$$\dot{\xi} = -a t_0 \xi + b t_0 x_0 \xi^2 + t_0 x_0^3 \xi^4$$

now figure out parameters... $+a t_0 = 1$ then $t_0 = 1/a$

$b t_0 x_0 = 1$ then $x_0 = \frac{1}{t_0 b} = \frac{a}{b}$

this leaves $t_0 x_0^3 = \frac{1}{a}\left(\frac{a}{b}\right)^3 = \frac{a^2}{b^3} = \alpha$

New ODE $\qquad \dot{\xi} = -\xi + \xi^2 + \alpha \xi^4$ where $\alpha = \frac{a^2}{b^3}$

analyze this in terms of α as SINGLE parameter.