Today we want to analyze <u>NONLINEAR SYSTEMS</u>

$$\dot{x} = f_1(x,y)$$
$$\dot{y} = f_2(x,y)$$

PHASE PLANE is helpful ... but limited.

IMPORTANT IDEAS:

- FIXED POINTS — we always solve for these.
  $$f_1(x,y)=0$$
  $$f_2(x,y)=0$$
  STABLE, UNSTABLE, CENTER, ...
  sinc, source
- CLOSED ORBITS — orbits that correspond to oscillating or repeating solutions.
- THE BEHAVIOR OF FP ... type of fixed point and confirm stability.

Numerical solutions are possible — USE RUNGE KUTTA but only if a <u>NUMBER</u> is needed.

Helpful Idea: NULLCLINES — lines in the phase plane along which one of $\dot{x}$ or $\dot{y}$ is zero — flow in only one direction.

<u>Ex</u> $\dot{x} = x + e^{-y}$ FIXED POINTS $x + e^{-y} = 0$ $x + e^{-0} = x + 1 = 0$ $x = -1$

$\dot{y} = -y$ $-y = 0$ so $y = 0$

one FP at $(-1,0)$

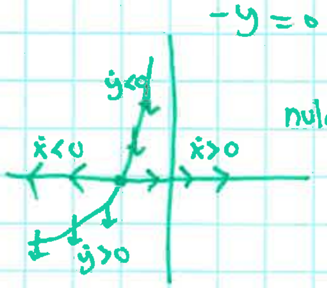
Phase Plane — This Appears to be a saddle point.

NULLCLINES: $x + e^{-y} = 0$ or $e^{-y} = -x$ $y = \ln\left(\frac{1}{-x}\right)$

$-y = 0$ or $y = 0$

along $y=0$
$\dot{x} = x+1$

along $y = \ln\left(\frac{1}{-x}\right)$
$\dot{y} = -y$



nullclines intersect @ Fixed Points   confirms <u>SADDLE</u>...

BUT... I WOULD LIKE MORE PROOF!

FIXED POINTS AND LINEARIZATION. — mathematical analysis of fixed points...

Goal: Find Fixed Points — zoom in on them to see what kind of FP's they are.

Consider the system $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

suppose that (x^*, y^*) is a fixed point.

We want to see what hapens right around this point
"flick it and see what it does"

let $u = x - x^*$ and $v = y - y^*$
u – flick in x-direction
v – flick in y-direction

Now we want to know do these disturbances grow or decay.

grow → unstable
decay → stable.

$$\dot{u} = \frac{d}{dt}(x - x^*) = \dot{x}_{\text{near } x^*, y^*} = f(x^*+u, y^*+u) = f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

Two-dimensional Taylor series

$$\dot{v} = \frac{d}{dt}(y - y^*) = \dot{y}_{\text{near } x^* y^*} = g(x^*+u, y^*+u) = g(x^*, y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

so
$$\dot{u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$
$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}$$

$$\Longrightarrow \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

this is called the Jacobian!

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x^*, y^*)}$$

Jacobian @ the Fixed Point.

Now we analyze $\dot{\vec{x}} = Ax$ where $x = \begin{bmatrix} u \\ v \end{bmatrix}$

using linear methods ... this will tell us about the fixed point.

NOTE – if this analysis results in a borderline case STAR, CENTER, DEGENERATE NODE we must be carefull – the small "neglected" nonlinear terms play a role.

<u>Ex</u> Analyze the nonlinear system

$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y.$$

1. Find all possible fixed points...

$-x + x^3 = 0 \quad x(-1 + x^2) = 0 \quad x = 0,\ x = -1,\ x = 1$

$-2y = 0 \Rightarrow y = 0$

Three Fixed Points $(0,0), (-1,0), (1,0)$

2. Calculate the Jacobian

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{bmatrix}$$

the jacobin can have functions inside!

3. NOW CONSIDER EACH FP INDIVIDUALLY...

a) FP : $(0,0)$ $\quad A = J\Big|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

$\tau = -3 \quad \Delta = 2 \quad \lambda = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$

$\lambda_1 = -1 \quad \lambda_2 = -2$

This is a <u>stable node</u>

b) FP: $(-1,0)$ $\quad A = J\Big|_{(-1,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$\tau = 0$ and $\Delta = -4$ $\quad \lambda = \frac{0 \pm \sqrt{0+16}}{2} = \frac{\pm 4}{2}$

$\lambda_1 = 2 \quad \lambda_2 = -2$

This is a saddle point.

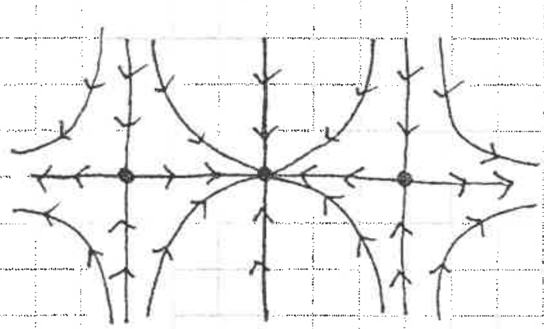
c) FP: $(1,0)$ $\quad A = J\Big|_{(-1,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ same as $(-1,0)$

This is a saddle point.

4. FIND NULLCLINES FROM ORIGINAL SYSTEM

$\dot{x} = -x + x^3 = 0 \quad x=0, \; x=1, \; x=-1$ $\qquad \dot{y} = -2y = 0 \quad y=0$

5. Phase Portrait.



#1 Draw Your FP's

#2 Draw the nullclines

#3 Decide on direction for nulclines.

Ex. For the $y=0$ nulcline only have flow in the x-direction so look at $\dot{x} = -x+x^3$ around FP to look for pos or neg flow

#4 Look @ what type of node it is – From Linearization analysis – to fill in the details

NOTE: you can always check phase portrait computer plot to make sure this is correct.

QUICKLY... Existence and Uniqreness ... generalized from the 1-D case.

Given $\dot{\vec{x}} = \vec{f}(\vec{x})$ $\quad \vec{x}(0) = x_0$

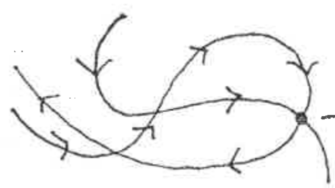
Suppose $\vec{f}$ is continuous and that all of its first partial derivatives are continuous $\frac{\partial f_i}{\partial x_j}$ $i,j = 1, \ldots, n$

on some open connected set $D \subset \mathbb{R}^n$. Then for $x_0 \in D$, the initial value problem has a soln on some time interval $(-\tau, \tau)$ about $t=0$ and the solution is unique.

5.

An Important Idea comes from this ...

If a solution exists and is unique $\rightarrow$ different trajectories never intersect

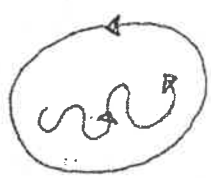


$\longrightarrow$ two solutions go through this point $\Rightarrow$ solution not unique, how could we choose the right one ?!?

Phase Portraits – always look "well-groomed"

OTHER COOL IDEAS ...

If we can find a closed orbit in our phase portrait then any trajectory starting inside must stay inside



Futher ... if there are no fixed points inside the closed orbit...

Intuition $\rightarrow$ the trajectory can't go forever w/o crossing!

POINCARE- BENDIXSON THEORM ... If a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory MUST approach a closed orbit.

BREAK ... plot phase portraits on pplane – start HW.

If we are interested in finding quantitative aspects — we would need to do numerical computation.

Always use Runge-Kutta : in vector form: for any higher order system

$$\vec{k}_1 = \vec{f}(\vec{x}_n)\Delta t$$
$$\vec{k}_2 = \vec{f}(\vec{x}_n + \tfrac{1}{2}\vec{k}_1)\Delta t$$
$$\vec{k}_3 = \vec{f}(\vec{x}_n + \tfrac{1}{2}\vec{k}_2)\Delta t$$
$$\vec{k}_4 = \vec{f}(\vec{x}_n + \vec{k}_3)\Delta t$$

Don't let the vectors fool you ... this is pretty much the same as 1-D.

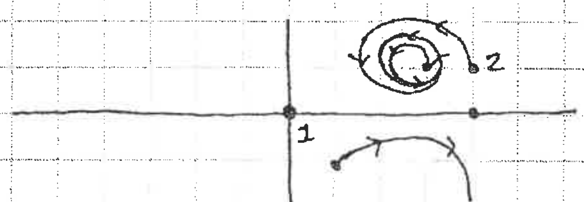
$$\vec{x}_{n+1} = \vec{x}_n + \tfrac{1}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$$

written in standard form for 2-D

$$x_{n+1} = x_n + \tfrac{1}{6}(k_{x_1} + 2k_{x_2} + 2k_{x_3} + k_{x_4})$$
$$y_{n+1} = y_n + \tfrac{1}{6}(k_{y_1} + 2k_{y_2} + 2k_{y_3} + k_{y_4})$$

$$\vec{k}_1 = \begin{bmatrix} k_{x_1} \\ k_{y_1} \end{bmatrix}$$

Then to get the phase portrait from this we must run the code for lots of different initial conditions



run 1 $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ nothing happens...

run 2 $\vec{x}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

run 3 $\vec{x}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

run 4 $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

after doing a bunch of these you will get a phase portrait.