Last Time: Linearization and Fixed Points for Nonlinear Systems.

1. Find $\longrightarrow$ Fixed Points — $\dot{x}=0$ AND $\dot{y}=0$
   Nullclines — $\dot{x}=0$ OR $\dot{y}=0$

2. Find the Jacobian:
$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

3. "Linearize" Now zoom in on each fixed point
   plug in each (x^*, y^*) $\quad J\Big|_{(x^*, y^*)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

   Analyze the linear system
   $\dot{\vec{u}} = A\vec{u}$ eigenvalues eigenvectors.

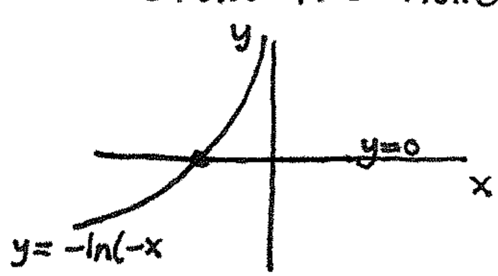
4. Sketch the phase portrait
5. Discuss the physical meaning of the solution.

Lets try this on the example from yesterday.

$\dot{x} = x + e^{-y}$
$\dot{y} = -y$

Fixed Points $\longrightarrow$ $x+e^{-y}=0$ and $-y=0$
Nullclines $\quad x+e^{-y}=0$ or $-y=0$

Note... draw the nullclines to find difficult fixed points:



$-x = +e^{-y} \longrightarrow \ln(-x) = -y$

$\boxed{y = -\ln(-x)}$

and $\boxed{y=0}$

The fixed point is where these cross: $(-1, 0)$ Matches what we found yesterday.

Analyze the stability of $(-1,0)$

$$J = \begin{bmatrix} 1 & -e^{-y} \\ 0 & -1 \end{bmatrix}$$

plug in the point $(-1,0)$

$$J\Big|_{(-1,0)} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues $\tau = 0$ $\Delta = -1$

$$\lambda = \frac{0 \pm \sqrt{0-4(-1)}}{2} = \pm 1$$

Should be a saddle.

Eigenvectors.

$\lambda = 1$ then $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 1-1 & -1 \\ 0 & -1-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

We can still use eigenvectors as directions right close to the fixed point.

$-1v_2 = 0$ $-2v_2 = 0$
all we find $v_2 = 0$
v_1 = anything.

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ works.

positive flow away from fp

$\lambda = -1$ then $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 1-(-1) & -1 \\ 0 & -1-(-1) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

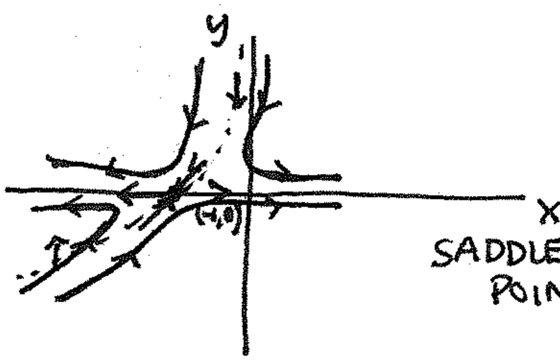
$2v_1 - v_2 = 0$

$v_2 = 2v_1$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ works.

negative flow.

Plot:



SADDLE POINT.

Manifolds are curved in nonlinear systems.

Doing this kind of Linearization is <u>GREAT</u>... why...

HW #1
Day 1

Just looking at pplane
Looks like a degenerate node
ONLY ONE MANIFOLD

<u>BUT</u> Check the stability of the FP...

$$\dot{x} = y \qquad (0,0) \qquad J = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$
$$\dot{y} = -2x - 3y$$

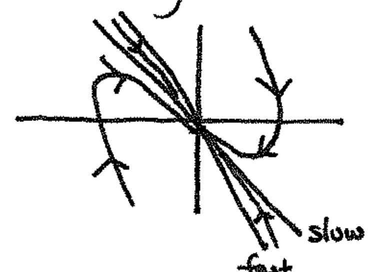
$$\tau = -3 \qquad \Delta = 2$$

$$\lambda = \frac{-3}{2} \pm \frac{\sqrt{9-8}}{2} = -\frac{3}{2} \pm \frac{1}{2}$$

This is a
STABLE NODE
TWO MANIFOLDS!

$$\lambda = -2, -1$$

They are just really close together!



Different behavior based on where you start!

<u>HOWEVER</u>... this sometimes fails!

<u>Ex</u>
$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

This system has a fixed point at the orgin $(0,0)$

Analyze the fixed point.

$$J = \begin{bmatrix} 3ax^2 + ay^2 & -1 + 2axy \\ 1 + 2ayx & ax^2 + 3ay^2 \end{bmatrix}$$

$$J\Big|_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{matrix} \tau = 0 \\ \Delta = 1 \end{matrix} \qquad \lambda = 0 \pm \tfrac{1}{2}\sqrt{-4} = \pm i$$

predicts centers!
doesn't mater what a is!

Pplane — If $a=0$ then we are right!

However... even a small change in a changes the pplane.

Totally changes the stability!

IF $Re\{\lambda\}=0$ then the linearization tells us nothing. because we are making a STRONG assumption that higher order terms don't matter!

Hints for drawing Phase Plane — If a solution exists and is unique then trajectories never cross!

this could never happen.

Only manifolds can cross!!

Then the soln is not unique.

<u>Another Example</u>:

The Brusselator –

$$\dot{x} = 1 - (1+b)x + ax^2y$$
$$\dot{y} = bx - ax^2y$$

Briggs – Rauscher hypothetical chemical oscillator.

x and y are the chemical concentrations.

a and b are parameters of the reaction. taken to be positive #'s.

Analyze this system and talk about the results.

Fixed points: $1-(1+b)x + ax^2y = 0$ and

$bx - ax^2y = 0$ then $y = \frac{b}{ax}$

$$1-(1+b)x + \frac{ax^2 b}{ax} = 0$$

$$1-(1+b)x + bx = 0$$

$1-x=0$ or $x=1$

so $y = \frac{b}{a}$

Only one fixed point.

Jacobian

$$J = \begin{bmatrix} -(1+b) + 2axy & ax^2 \\ b-2axy & ax^2 \end{bmatrix}$$

$$J\Big|_{1,\frac{b}{a}} = \begin{bmatrix} -1-b+2a(1)\frac{b}{a} & a \\ b-2a(1)\frac{b}{a} & -a \end{bmatrix} = \begin{bmatrix} b-1 & a \\ -b & -a \end{bmatrix}$$

$\tau = a+b-1 \qquad \Delta = -a(b-1) + ab = a$

so

$$\lambda = \tfrac{1}{2}(a+b-1) \pm \tfrac{1}{2}\sqrt{(b-a-1) - 4a}$$

because $a>0$ $\Delta > 0$ so we never have saddles.

<u>But if</u> $(b-a-1) - 4a < 0$ then we get oscillations.

$b-a-1 < 4a \qquad b < 4a+a+1$

$b < 5a+1$

<u>And if</u> $b-a-1<0$ Stable $b<a+1$

$b-a-1>0$ Unstable $b>a+1$

Cases... $a=b=1$ $1<1+1$ Stable

$1<5+1$ Oscillations.

Thes solution $\rightarrow$ to the fixed point the concentrations settle down.

$a=1$ $b=3$ $3>1+1$ Unstable

$3<5+1$ Oscillations.

The soln goes away from the fixed point look at P-plane.

OSCILLATIONS.
LIMIT CYCLE.