Two Dimensional Bifurcations.

Saddle Node
Transcritical
Pitchfork

} These still happen in higher dimensional systems... along manifolds.

NEW → Now we can have the creation of OSCILLATIONS or stable → unstable spirals.

HOPF BIFURCATION.

SADDLE NODE – in 2-D.

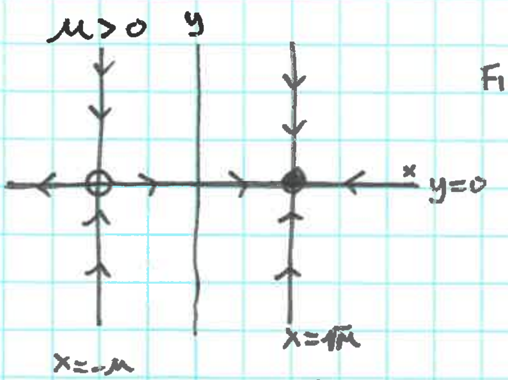
normal form

$$\dot{x} = \mu - x^2 \qquad \text{FP: } \mu - x^2 = 0 \qquad x = \pm\sqrt{\mu}$$
$$\dot{y} = -y \qquad\qquad y = 0$$

bifurcation when $\mu = 0$

$\mu > 0$ $\mu < 0$

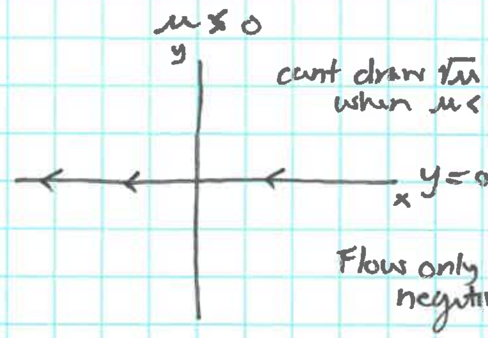
Nullclines: $y = 0$ (Blue)
$x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ (red.



Fixedpoints where nullclines intersect.

as $\mu \to 0$ the FP approach.

TWO FP $(-\sqrt{\mu}, 0)$ $(\sqrt{\mu}, 0)$



can't draw $\sqrt{\mu}$ when $\mu < 0$

Flows only negative.

NO FP

Bottleneck Region – When the FP meet and annihilate they leave behind a GHOST – it sucks in trajectories and slows them down before they leave the other side!

SHOW PPLANE.

Consider Jacobian Analysis. $J = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda = -2x$ $\lambda = -1$

$(-\sqrt{\mu}, 0)$ $\lambda = 2\sqrt{\mu}, -1$ SADDLE
$(\sqrt{\mu}, 0)$ $\lambda = -2\sqrt{\mu}, -1$ STABLE NODE

$\lambda_1 = \pm 2\sqrt{\mu}$ $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda_2 = -1$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Always stable Manifold.

2-D Saddle Node — Fixed points move toward each other along an unstable manifold, meet, and annihilate.

OR ... are created and move away along unstable manifold.

Still a Fundamentally 1-D process.

LOOK AT NULLCLINES TO SEE THIS BEHAVIOR...

Example Genetic Control System.

$\dot{x} = -ax + y$ protein concentration

$\dot{y} = \dfrac{x^2}{1+x^2} - by$ messenger RNA concentration

$a, b > 0$ rate of degradation

* Gene encodes protein but its own activity is stimulated by that protein.

Nonlinear Analysis:

FIXED POINTS and NULL CLINES

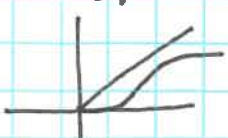
$-ax^* + y = 0$ so $y = ax$

$\dfrac{x^2}{1+x^2} - by = 0$ so $y = \dfrac{x^2}{b(1+x^2)}$

Fixed points where these intersect!

CASES.

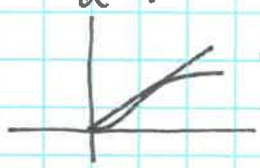
a bigger $a > .5$



ONE FP $(0,0)$

$x=0, y=0$ always an FP.

$a \sim .5$



lines Touch.

TWO FP ?

$a < .5$ smaller

THREE FP

Two FP created

NOTE... changes in b do a similar thing!!

Solve for FP...
set nullclines equal.

$(0,0)$ is ONE FP....

$ax = \dfrac{x^2}{b(1+x^2)}$. $a = \dfrac{x}{b(1+x^2)}$ $ab + abx^2 = x$ $abx^2 - x + ab = 0$ quadratic eqn!

$$x = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$$

when $1-4a^2b^2 < 0$ ONE FP

$1-4a^2b^2 = 0$ Bifurcation occurs.

$1-4a^2b^2 > 0$ THREE FP.

and $y = ax$ to solve for the other coord.

What is the critical value? $1-4a^2b^2=0$

$a^2b^2=\frac{1}{4}$ or $\boxed{ab=\frac{1}{2}}$ positive since $a,b>0$

a bifurcation occurs when

Jacobian: $J=\begin{bmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{bmatrix}$

Trace is Easy!

in general before FP $\tau=-(a+b)<0$ always.

so SADDLE $\Delta<0$ STABLE SPI. NODE $\Delta>0$

Determinant more complicated...

$$\Delta = ab - \frac{2x^*}{(1+x^{*2})^2}$$ must plug in FP to see...

$(0,0)$ $\Delta\Big|_{(0,0)} = ab>0$ the origin is STABLE.

$$\lambda = \frac{-(a+b)\pm\sqrt{(a+b)^2-4ab}}{2} = \frac{-a-b\pm(a-b)}{2}$$

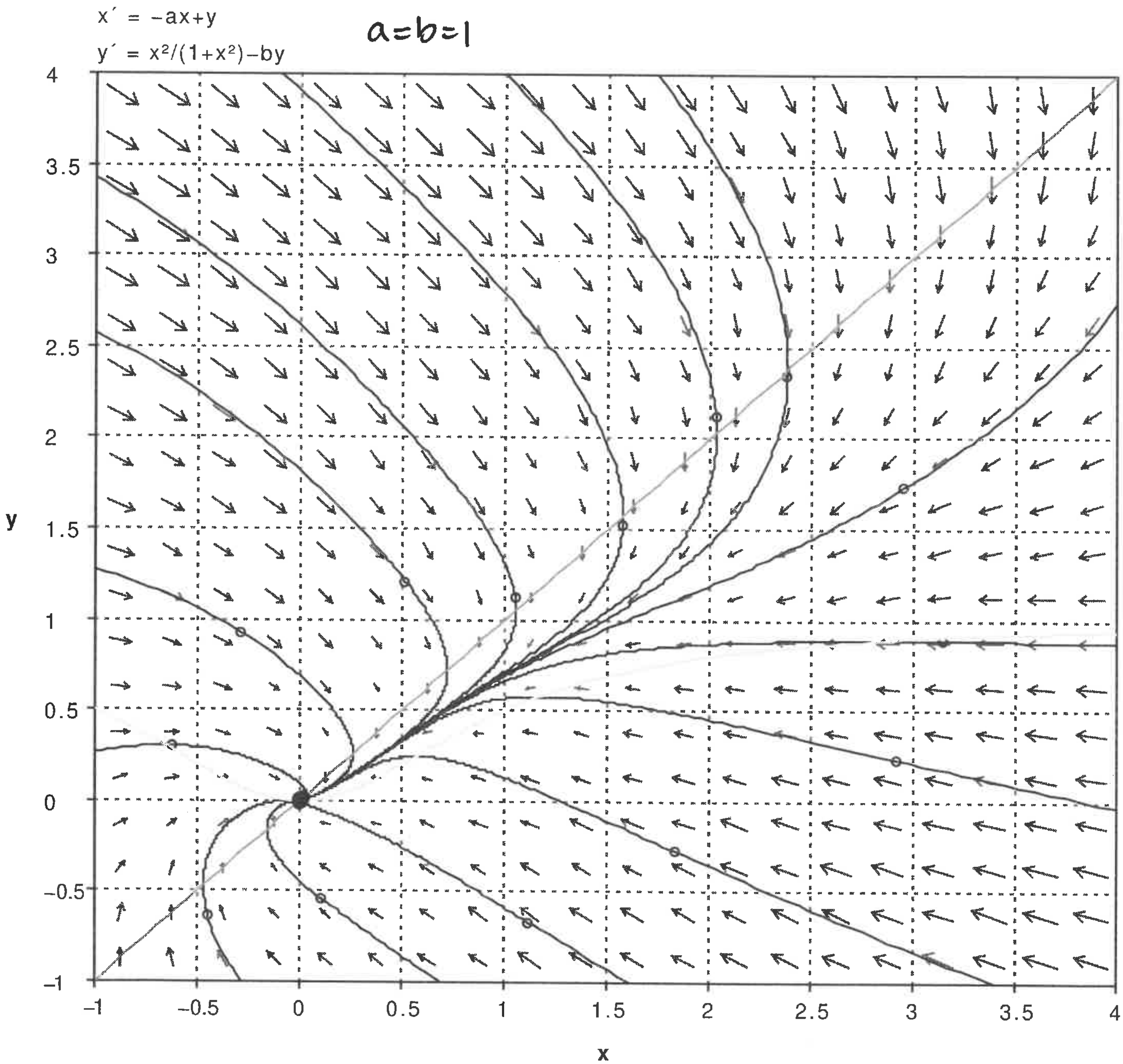
so $\lambda=-b,-a$

ALWAYS A STABLE NODE.

$\left(\frac{1\pm\sqrt{1-4a^2b^2}}{2ab}, \frac{1\pm\sqrt{1-4a^2b^2}}{2b}\right)$ plug this in! A REAL MESS...

LOOK AT PHASE PLANE! $a=b=1$ and $a=\frac{1}{4}$ $b=1$ [BF point. $a=\frac{1}{2}$ $b=1$]

$a=b=1$



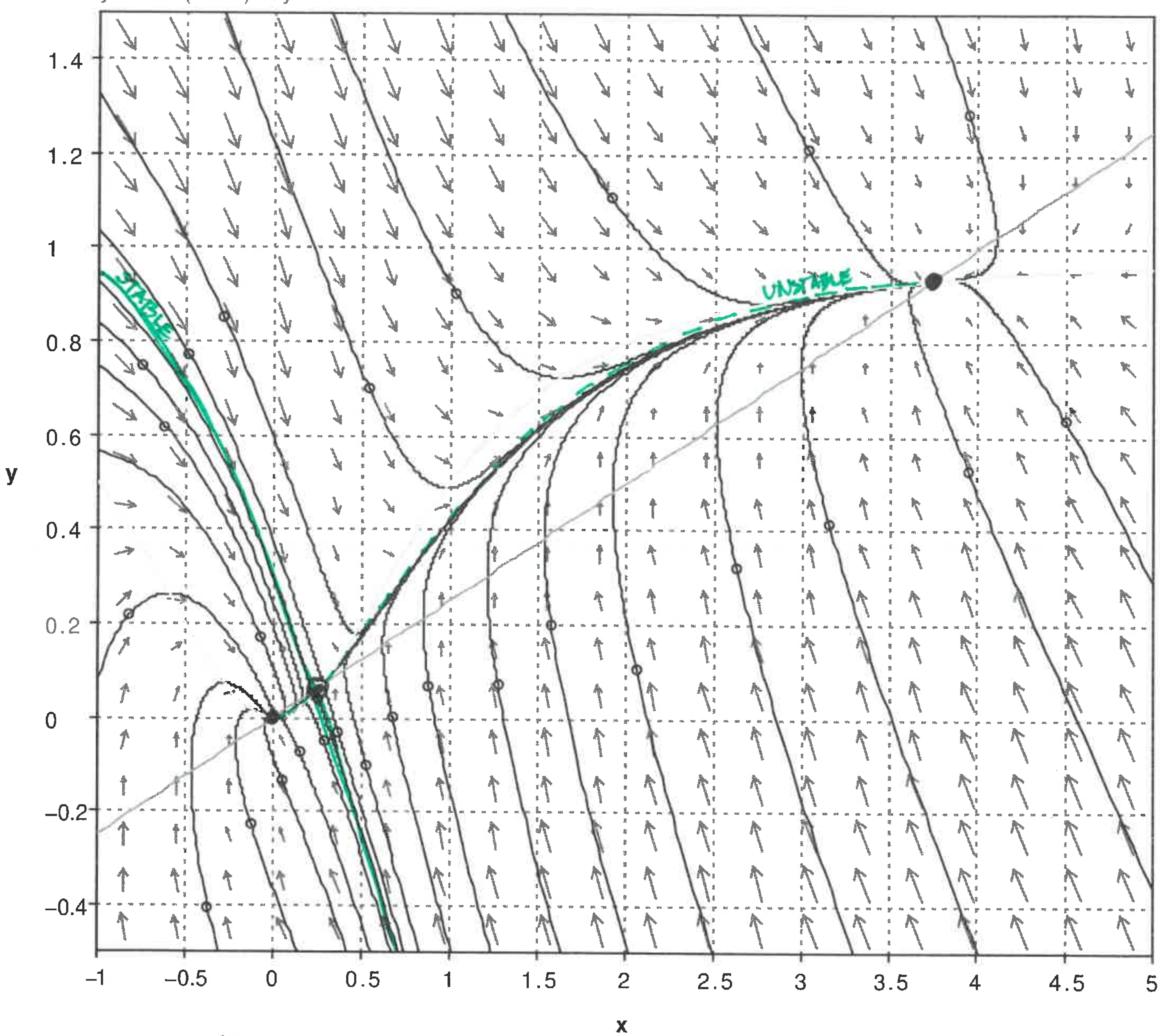
one stable fixed point at the orgin —
$ab > \frac{1}{2}$ degradation is large
so both concentrations go to zero!

protein and RND decay slowly enough that there is a chance for GENE EXPRESSION,

$x' = -ax+y$

$y' = x^2/(1+x^2)-by$

$a=\frac{1}{4}$ $b=1$



THREE FP!

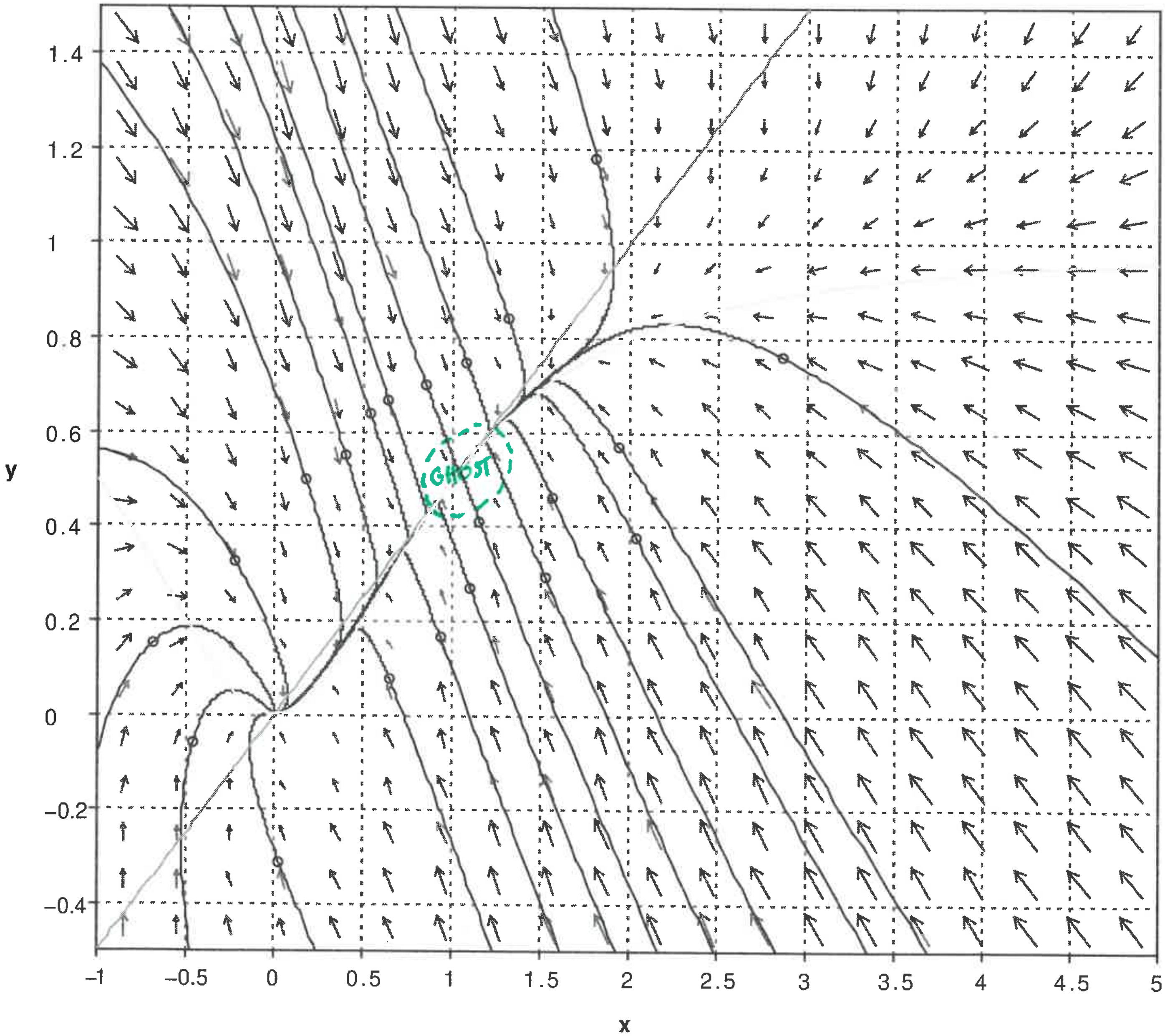
(0,0) STABLE then along the unstable manifold it goes SADDLE UNSTABLE then STABLE NODE.

IF we start ABOVE THE LINE ——— Stable manifold. we are in the basin of attraction For non-zero FP.

otherwise we go to (0,0).

$x' = -ax+y$
$y' = x^2/(1+x^2)-by$

$a=\frac{1}{2} \quad b=1$



When the bifurcation occurs the FP meet along the unstable manifold and annihilate!

They create a GHOST... a region that attracts and slows down trajectories before letting them go to the stable FP.

From Last Time:

Brusselator: $\dot{x} = 1-(1+b)x + ax^2y$
$\dot{y} = bx - ax^2y$

we found: $x^*=1 \quad y^* = b/a$
OSCILLATIONS when $(b-a-1)-4a<0$
when $b<a+1$ STABLE
$b>a+1$ UNSTABLE

A change in the dynamics of the system!

We need to develop a new kind of bifurcation for this!!

$a=1 \quad b=3$ spiral is unstable but it approaches a LOOP $\longrightarrow$ $a=1 \quad b=2.1$ The LOOP is closing in on the FP $\longrightarrow$ $a=1 \quad b=1$ The loop disappears The FP is stable spiral

FIRST... UNDERSTAND THE LOOP.

Limit Cycles are closed trajectories in the phase plane

1. Stable or Attracting – all nearby trajectories approach the limit cycle.
2. Unstable – all nearby trajectories move away from the limit cycle.
3. Half Stable – special case one side goes toward the other away.

These are difficult to Prove!

WE CAN.

PROVE SOME SYSTEMS HAVE NO CLOSED ORBIT:

Because $\nabla \times \nabla f = 0$ the curl of the gradient $=0$ if we can prove our system can be writen ∇f then there is no "curl" and no closed orbit.

What is ∇f ...

if $f(x) = x^2y$ then $\nabla f = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] f = \left[f_x, f_y\right] = \left[2xy, x^2\right]$

it is a vector.

OUR GOAL ... work backward.

Ex. Show that $\dot{x} = \sin y$
$\dot{y} = x\cos y$
has NO CLOSED ORBITS — no limit cycles.

Can we construct V such that $-\nabla V = \langle \dot{x}, \dot{y} \rangle$ potential field.

lets try... $-\frac{\partial V}{\partial x} = \dot{x} = \sin y$ $\qquad$ $-V = x\sin y + F(y)$

integrate

$-\frac{\partial V}{\partial y} = x\cos y$ $\qquad$ $-V = x\sin y + G(x)$

YES we found one $V(x) = -x\sin y$

So there is no curl in our system — No limit cycles.

Ex Brusselator $\dot{x} = 1-(1+b)x + ax^2y$
$\dot{y} = bx - ax^2y$

try.. $-\frac{\partial V}{\partial x} = 1+(1+b)x + ax^2y$ $\qquad$ $-V = x + \frac{1}{2}(1+b)x^2 + \frac{ax^3y}{3} + F(y)$

so

$-\frac{\partial V}{\partial y} = bx - ax^2y$ $\qquad$ $-V = bxy - \frac{ax^2y^2}{2} + G(x)$

These don't match in any way!
Can't find a $V(x)$. so $\dot{\vec{x}} = -\nabla V$

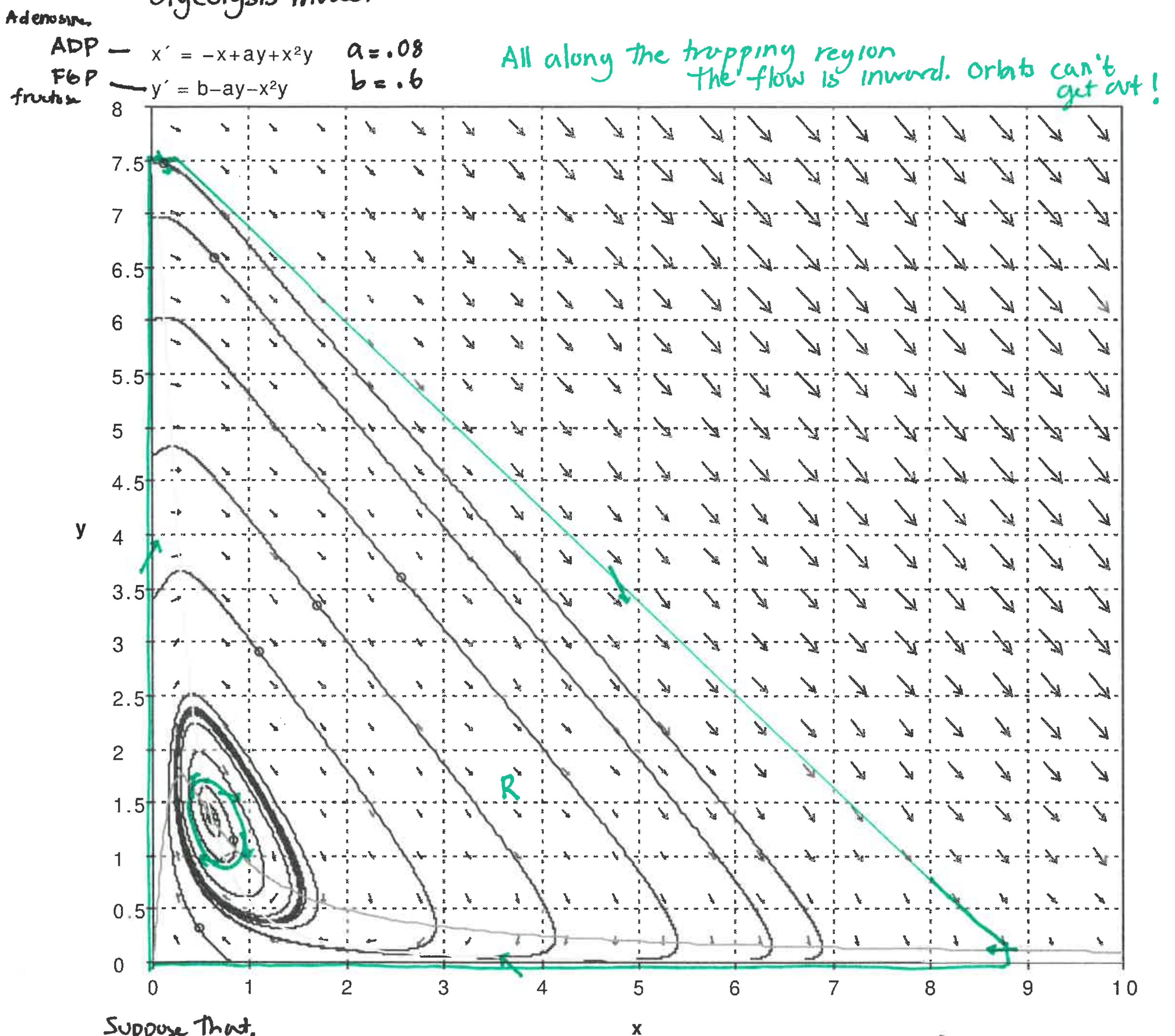
But this just means that a limit cycle might exist not that it DOES EXIST.

To establish that closed orbits exist: POINCARE-BENDIXSON THM!
SHOW PPLANE $\qquad$ only true in 2-D

NOTES 3-D systems can have trajectories "wander" around forever in a bounded region! — Strange Attractor / Fractal Set w/ Chaos.

No Chaos in continuous 2-D system

Glycolysis model



Suppose that.

(1) R is a closed, bounded subset of the plane

(2) $\dot{\vec{x}} = \vec{f}(\vec{x})$ is a continuously differentiable vector field on an open set containing R

(3) R does not contain any fixed points.

} Easy to satisfy.

(4) There exists a trajectory C that is confined in R. → Hard to show. Construct a Trapping Region.

Then either C is a closed orbit or it spirals toward a closed orbit as $t \to \infty$