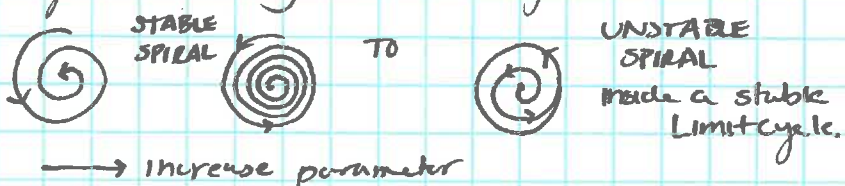
IN 2 and higher dimensions we can add many different kinds of bifurcations!

FAMOUS ONE: Hopf Bifurcation: spirals go from stable to unstable.

solns go from decay to limit cycle oscillations.



Helps to study the onset of spontaneous oscillations.

<u>Ex</u> $\dot{x} = y + mx$

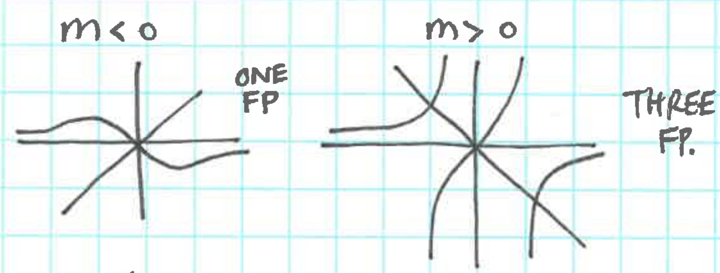
$\dot{y} = -x + my - x^2y$

<u>Nullclines</u> and <u>Fixed Points.</u>

$\dot{x}=0 \updownarrow$ $\quad$ $\dot{y}=0 \leftrightarrow$

$y=-mx$ $\quad$ $y = \dfrac{x}{m-x^2}$

ON DESMOS



<u>Solve for FP...</u>

(0,0) is always a Fixed Point!

$$-mx = \frac{x}{m-x^2} \rightarrow -m = \frac{1}{m-x^2}$$

m=0 Bifurcation point.

$$\rightarrow -m^2+mx^2=1 \rightarrow x = \pm\sqrt{\frac{1+m^2}{m}} \rightarrow \text{problems } m=0$$

match.

when $m<0$ (0,0) is our only FP

when $m>0$ (0,0) and $\left(\sqrt{\frac{1+m^2}{m}}, -m\sqrt{\frac{1+m^2}{m}}\right)\left(-\sqrt{\frac{1+m^2}{m}}, m\sqrt{\frac{1+m^2}{m}}\right)$

Three FP.

Jacobian: $J = \begin{bmatrix} m & 1 \\ -1-2xy & m-x^2 \end{bmatrix}$

$$J\Big|_{(0,0)} = \begin{bmatrix} m & 1 \\ -1 & m \end{bmatrix} \qquad \lambda = \frac{2m \pm\sqrt{4m^2-4(m^2+1)}}{2} = m \pm i$$

SPIRALS.

$\tau = 2m \quad \Delta = m^2+1$

$m<0$ stable $\quad$ $m>0$ unstable.

Bifurcation causes spiral to become unstable.

when $m > 0$

$$J\Big|_{x=\pm\sqrt{\frac{1+m^2}{m}}} = \begin{bmatrix} m & 1 \\ 1-7m^2 & -\frac{1}{m^2} \end{bmatrix}$$

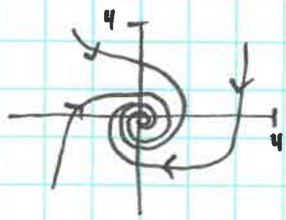
messy...

to find $\lambda = \frac{m^2-1}{2m^2} \pm \frac{1}{2}\sqrt{\left[\frac{m^2-1}{m^2}\right]^2 + 4\left[\frac{m-1}{m}\right] - 8m^2}$

when $m > 0$
we have on pos and one neg.
These are SADDLES!

PHASE PLANE

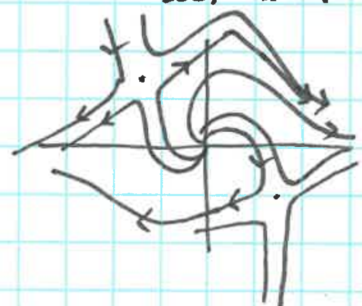
$m < 0$ let $m = -1$ ⟶ $m > 0$ let $m = 1$



All solns decay to (0,0)

STABLE SPIRAL

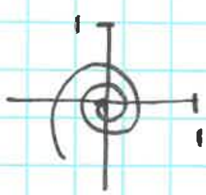
But something interesting happens near $m = 0$



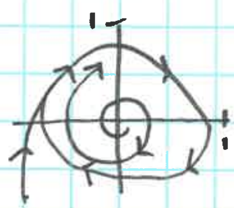
TWO SADDLES AND AN UNSTABLE SPIRAL.

ZOOM IN TO (0,0)

$m = -.1$



$m = .1$



ORIGIN IS UNSTABLE
BUT LIMIT CYCLE.

when m is very small — zero — linear stability $\Rightarrow$ centers.

$$\dot{x} = y + mx \qquad\qquad \dot{x} = y$$
$$\dot{y} = -x + my - x^2 y \quad \sim \quad \dot{y} = -x - x^2 y$$

this term is important can't be neglected.

A true HOPF bifurcation has limit cycle oscillations near the bifurcation.

STABLE ⟶ OSCILLATIONS ⟶ UNSTABLE.

Going from Bifurcations to Chaos to Fractals... There is so much more to study!!

Start this Story with the LORENZ EQNS.

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

A simplification "first order" of the NAVIER STOKES eqns.

Fluid heated from below...

HERE σ = Prandtl #
r = Rayleigh # — How much HEAT
b = aspect ratio of fluid shed.

x = convective intensity
y = temperature
z = deviation from linearity (vertical)

Edward Lorenz — Deterministic Nonperiodic Flow (1963)

Plot phase trajectories — 3D — for different values of r
let $\sigma = 10.5$ and $b = 1$

r — small little heating. $r=13$ TWO STABLE NODES... can't tell which one soln will choose.

- Highly sensitive to INITIAL CONDITIONS

As r is increased a bifurcation occurs.
We lose our two stable spirals and get a curve in space that solns are Attracted to.

$r=13$ vs $r=14$ vs $r=14.3$ ... $r=18.5$ ... $r=30$
two curves are intertwined.
very complicated shape!

CHAOS A-periodic long term behavior... solns don't find fixed points or limit cycle orbits.
Deterministic — no randomness
Sensitive to Initial Conditions — trajectories separate exponentially fast.

STRANGE ATTRACTOR — this curve in phase space is very complicated. "An infinite complex of surfaces" To describe this set we need fractal geometry!

Introduction to Fractals:

Many objects look the same at different scales especially natural objects.

Ex Leaves, branches, trees
Cauliflower
Small rock ... huge mountain

→ this idea is used to generate scenes in CGI Movies.

Fractals are shapes that have a non-integer dimension — ROUGH SHAPES.

Benoit Mandelbrot (1970's) called this idea of "self similar" objects = fractals. → calculus — things look smooth when you zoom it. Fractals allow for Roughness.

Fractals — Don't HAVE to be self similar.

Interesting BOTH artistically and scientifically.

To see what I mean... lets draw a fractal:

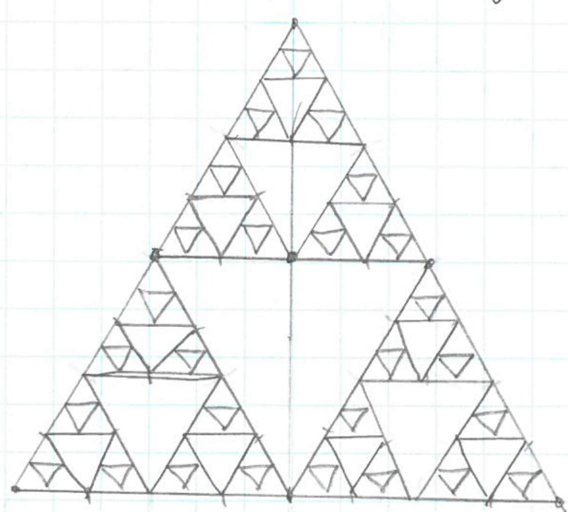
Example of Fractle Von Koch Snowflake.

• Divide each line into 3
• Put a triangle in middle section

Keep going Forever

Examples of fractals: Sierpinski triangle:

start by constructing an equilateral triangle.

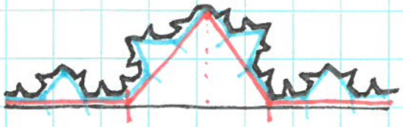


• Divide each segment in $1/2$
• connect the dots.

outer triangles only

Keep going Forever.

Von Koch Snowflake



Start w/ Line
Divide Line into 3
Draw triangle on middle $\frac{1}{3}$
Divide those lines into 3
Draw triangle on middle $\frac{1}{3}$

Again and Again.

Sierpinski Triangle

Start w/ Equilateral Triangle
Divide each side in $\frac{1}{2}$
Connect lines to draw interior triangle
Divide all lines in $\frac{1}{2}$
Connect to draw interior triangles
~~even~~ ON OUTER SIDES.

Again and again

Many examples like this exist... some fractals you can draw by hand others are more complicated!

Mandelbrot set $\quad f_c(z) = z^2 + c$

c = complex # $\quad c = a + bi$

Iterate on #0 to see what happens.

Makes a beautiful and complicated shape

Black — all values of c so the iterations converge

Color — values of c that diverge (BLOW UP)
color depends on speed of Blow up.

FRACTAL DIMENSION. – best way to deal with hard ideas is to consider their relationship with easier ideas!

ONE DIMENSIONAL

cut lengths in $\frac{1}{2}$

weight of each piece?

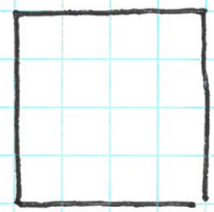
$\frac{1}{2}$ of original.

HERE

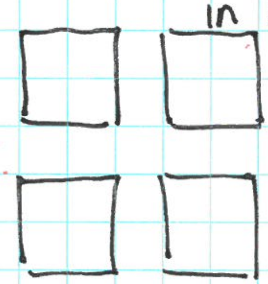
$$\underset{\text{lengths}}{\frac{1}{2}} \Rightarrow \underset{\text{mass}}{\frac{1}{2}} = \left(\frac{1}{2}\right)^1$$

Imagine that these things have mass or weight.

TWO DIMENSIONAL



cut lengths in $\frac{1}{2}$

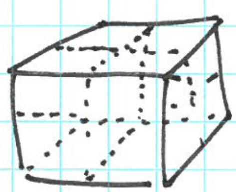


weight of each piece $\frac{1}{4}$ of original.

HERE

$$\underset{\text{lengths}}{\frac{1}{2}} \Rightarrow \underset{\text{mass}}{\frac{1}{4}} = \left(\frac{1}{2}\right)^2$$

What is this in THREE DIMENSIONS?



→ eight smaller cubes

$$\underset{\text{lengths}}{\frac{1}{2}} \Rightarrow \underset{\text{mass}}{\frac{1}{8}} = \left(\frac{1}{2}\right)^3$$

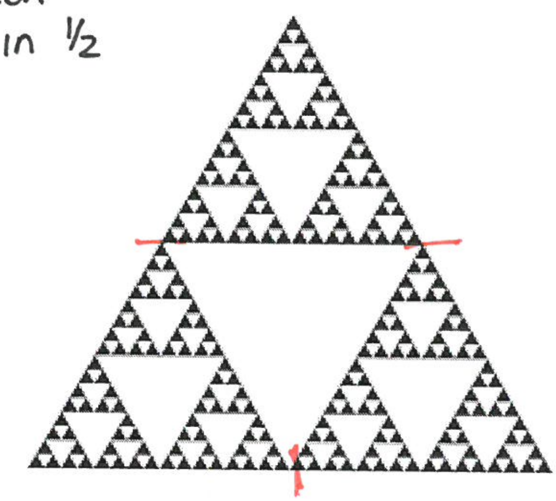
so what is the relationship between length, mass, and dimension?

$$\left(\frac{1}{2}\right)^D = \text{mass} \quad ✓$$

$$(\text{length})^D = \text{mass}$$

What if we did the same thing to a fractal...

Cut each length in ½



You get a copy of the original...

But what would be the change in weight or mass?

mass = ⅓ of original.

Using our formula:

$$\left(\frac{1}{2}\right)^D = \frac{1}{3}$$ we can solve for the dimension!

$$2^D = 3$$

or $D = \log_2(3) = 1.584963$

So we say this has dimension 1.584963 !!

Coast of Ireland = 1.22
England = 1.25
Norway = 1.52

Gives some measure of how rough the coast really is...

Surface of Human Brain = 2.79